LECTURE 2: TORIC VARIETIES

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Recall

- A symplectic manifold is an even dimensional manifold M^{2n} with a closed 2-form $\omega \in \Omega^2(M)$ such that $\omega^n{}_{|x} \neq 0 \ \forall x$.
- We assume that S^1 acts on M.
- We also assume the existence of a moment map for this action, that is $\phi : M \longrightarrow \mathbb{R}$ defined by $\imath_{\xi_M}\omega = d\phi$. We say $x \in M$ is critical if $d\phi_{|x} = 0$. If we denote by M^{S^1} the set of fixed points of the action, then by definition of the moment map, x is critical if and only if x is a fixed point.
- The extrema of ϕ are critical, and hence fixed points. In particular, if M is compact then ϕ must have fixed points.
- Also covered were the equivariant Darboux theorem and symplectic reduction.

1 Extending the S^1 action to more general groups

Let a torus $T = (S^1)^k$ act on (M, ω). We can extend the moment map definition from the previous lecture to this action.

Definition *Denote by $\mathfrak{t}$ the Lie algebra of the torus T, and by $\mathfrak{t}^*$ its dual. Define ξ_M to be the vector field on M induced by the flow* $\exp(t\xi)$*. Then a map $\phi : M \longrightarrow \mathfrak{t}^*$ is a* moment map *if*

$$\imath_{\xi_M}\omega = -d\phi^\xi \quad \forall \xi \in \mathfrak{t}\,,$$

where ϕ^ξ is the component of ϕ in the ξ direction, i.e. $\phi^\xi(x) = \langle \phi(x), \xi \rangle$.

Note This definition reduces to the one from the previous lecture when $T = S^1$.

Example *The n-dimensional torus $(S^1)^n$ acts on $\mathbb{C}^n$ by $\lambda \cdot x = (\lambda_1 x_1, \ldots, \lambda_n x_n)$.*

The moment map $\phi : \mathbb{C}^n \longrightarrow (\mathbb{R}^n)^*$ is

$$\phi(z) = \left(\frac{1}{2}|z_1|^2, \ldots, \frac{1}{2}|z_n|^2 \right) .$$

Note that $\phi(\mathbb{C}^n) = \left(\mathbb{R}^n_{\geq 0}\right)^*$, the image of the moment map is the positive orthant.

Claim ϕ is T-invariant and $\omega(\xi_M, \eta_M) = 0$ for all $\xi, \eta \in \mathfrak{t}$.

Exercise Prove the claim for compact M. *Hint*: Use the existence of fixed points of ϕ.

Definition *A T-action is* effective *if $T \longrightarrow \mathrm{Symp}(M, \omega)$ is injective, or in other words, if the identity element of T is the only element of T that fixes the whole of M.*

Note We will assume that all actions are effective.

Corollary (to the claim)

$$\dim T \leq \frac{1}{2} \dim M$$

Reason For any two vectors ξ, η in the torus direction, $\omega(\xi, \eta) = 0$ (no two vectors of T are paired under ω).

Theorem (Guillemin-Sternberg, Atiyah, 1982)

Suppose (M, ω) is a compact symplectic manifold with a moment map ϕ. Then

1. *$\phi^{-1}(a)$ is connected in $\mathfrak{t}^*$;*
2. *$\phi(M)$ is a convex polytope, in fact the convex hull of the images of the fixed points* $\mathrm{convhull}\big(\phi(M^T)\big)$.

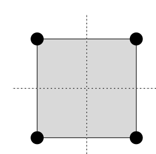
Note Since M is compact, there will be only finitely many connected components of fixed points. If we pick a connected component F of the set of fixed points M^T, then $\phi_{|F}$ is constant, because $d\phi = 0$ on F. So every component maps to a single point under the moment map. This is why the convex polytope $\phi(M)$ has finitely many vertices.

Note $\mathfrak{t}^* \cong (\mathbb{R}^n)^*$ since $\mathrm{Lie}(S^1) \cong \mathbb{R}$.

Example *Let the torus $T = S^1 \times S^1$ act on symplectic manifold $M = S^2 \times S^2$ by rotation in each fiber. Then the fixed points are $M^T = \{(N, N), (N, S), (S, N), (S, S)\}$ if N and S are the North and South poles of S^2. If the rotations are around the z-axes, the moment map is* (exercise)

$$\phi((x_1, y_1, z_1), (x_2, y_2, z_2)) = (z_1, z_2)\,,$$

and the image of M under the moment map is $\phi(M) = [-1, 1] \times [-1, 1]$.



So the fixed points here are the vertices of the convex hull. This isn't always true, but we'll see that it is for toric varieties.

2 Toric varieties

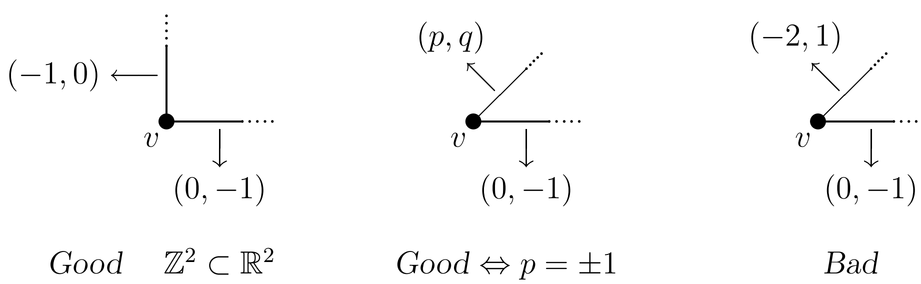
We want a class of manifolds for which the image of the moment map determines everything. For this, we want the action to be "big enough", and so we require that $\dim T = \frac{1}{2} \dim M$.

Definition *A* toric variety *is a compact connected $2n$-dimensional symplectic manifold M with an n-dimensional torus T action and a moment map $\phi : M \longrightarrow \mathfrak{t}^*$.*

Definition *A (n-dimensional)* Delzant polytope *is a polytope such that each vertex is contained in exactly n facets, and where the normals to the n facets containing a given vertex form a $\mathbb{Z}$-basis for a lattice $\mathfrak{l} \subset \mathfrak{t}$, so that $T = \mathfrak{t}/\mathfrak{l}$.*

<u>Fact</u> Let (M, ω, ϕ) be a toric variety. Then $\phi(M)$ is a Delzant polytope.

Example



The normals of the n facets containing v will be the $\mathbb{Z}$-basis of lattice if there is a transformation of $SL(n, \mathbb{Z})$ that sends them to the standard basis.

Exercise Show that the only Delzant polytope in $\mathbb{R}^n$ with $n+1$ facets is

$$\Delta = \{x \in \mathbb{R}^n : \ x_i \geq 0 \text{ and} \sum x_i \leq c\} \qquad (\text{some } c \in \mathbb{R}_{>0}),$$

up to translations and transformations of $SL(n, \mathbb{Z})$.

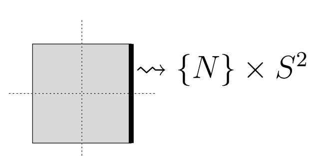
Theorem (Delzant)

1. *There is a one-to-one correspondence between toric varieties up to equivariant symplectomorphism and Delzant polytopes up to translation.*

2. *There is also a one-to-one correspondence between toric varieties up to equivariant symplectomorphism and automorphisms of T, and Delzant polytopes up to translation and transformations of $SL(n, \mathbb{Z})$.*

The Delzant polytope associated to a toric variety is determined by the moment map. Given a Delzant polytope, the associated toric variety is constructed via symplectic reduction of actions of subgroups of $(S^1)^k$ on $\mathbb{C}^k$, using the theorem of Darboux.

<u>Fact</u> In the case of a toric variety, the image $\phi(M^T)$ of the fixed points under the moment map are the vertices of the polytope $\Delta = \phi(M)$, and each edge corresponds to points with codimension 1 stabilizers.



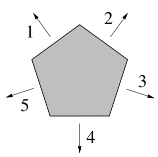
Thus the Delzant polytope contains all the important information about toric varieties; the fixed points and their images under the moment map give everything.

Here for example is how we would compute the cohomology ring $H^*(M_\Delta)$ of the toric variety M_Δ associated to a Delzant polytope Δ.

Suppose $\Delta = \{x \in \mathfrak{t}^* : \langle \eta_i, x_i \rangle \leq c\}$, where Δ has k facets $D_1, \ldots, D_k$ and $\eta_1, \ldots, \eta_k$ are the outward normals to these facets. First construct the set Σ containing, for all subsets of the facets that have a non-empty intersection, the set of indices of these facets:

$$\Sigma = \Big\{ I \subseteq \{1, 2, \ldots, k\} : \bigcap_{j \in I} D_j \neq \emptyset \Big\} .$$

For example, we would have



$$\Sigma = \big\{ \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,5\} \big\}.$$

We can then define the Stanley-Reisner ideal (due to Danilov)

$$J = \Big\{ \prod_{i_k \in I} x_{i_k} : I \notin \Sigma \Big\} .$$

In the pentagon example, J would contain $x_1 x_3, x_1 x_4$, etc.

We also need to define a second ideal, of linear relations:

$$K = \Big\{ \sum_i \langle \eta_i, \xi \rangle x_i : \xi \in \mathfrak{t} \Big\} .$$

With these definition, the cohomology ring of M_Δ is

$$H^*(M_\Delta) = \mathbb{C}[x_1, \ldots, x_k] \big/ (J + K) .$$

Example

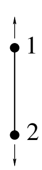


Image of the moment map on $\mathbb{C}\mathbb{P}^1$ *(or* S^2*)*

$$\mathbb{C}[x_1, x_2] \big/ \langle x_1 - x_2, x_1 x_2 \rangle \cong \mathbb{C}[x] \big/ \langle x^2 \rangle$$

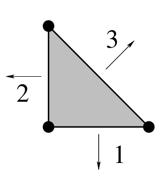
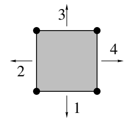
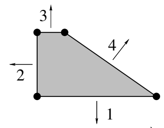


Image of the moment map on $\mathbb{C}\mathbb{P}^2$

$$\mathbb{C}[x_1, x_2, x_3] \big/ \langle x_1 - x_2, x_2 - x_3, x_1 x_2 x_3 \rangle \cong \mathbb{C}[x] \big/ \langle x^3 \rangle$$

Exercise Do this for 

and 

(where the slope of the diagonal side is n). The latter is called the *Hirzebruch n-surface).*

Note The first Chern class is $c_1(M) = \sum x_i$. In general, Chern classes are symmetric polynomials.

In the construction above, if instead of computing the (ordinary) cohomology, we want to compute the equivariant cohomology, we quotient the polynomial ring by J only.

Discussion

Charney-Davis conjecture

Consider an even dimensional simplicial polytope P and its associated toric variety X_P. Suppose that the dimension of P is $2e$. The boundary ∂P of P is a simplicial complex, and Danilov showed that the Betti numbers of X_P can be related to the h-vector of ∂P by

$$\beta_{2i}(X_P) = h_i(\partial P)\,.$$

It is conjectured that if the Stanley-Reisner ring of P is generated by quadratic monomials, then

$$(-1)^e \sum_{i=0}^{2e} (-1)^i h_i(\partial P) \geq 0\,.$$

The conjecture has a more general form: suppose Δ is a Gorenstein* simplicial complex, meaning that for every face F of Δ, the reduced homology of the link $\mathrm{lk}F$ of F is given by

$$\tilde{H}_i(\mathrm{lk}F) \cong \begin{cases} \mathbb{Z} & \text{if } \dim(\mathrm{lk}F) = i, \\ 0 & \text{otherwise.} \end{cases}$$

The general conjecture is that if Δ has dimension $2e-1$, its h-vector is $h(\Delta) = (h_0, h_1, \ldots, h_{2e})$ and its Stanley-Reisner ring is generated by quadratic monomials, then

$$(-1)^e \sum_{i=0}^{2e} (-1)^i h_i(\Delta) \geq 0\,.$$

Note If Δ has even dimension $2e$, the Dehn-Sommerville equations $h_i = h_{2e+1-i}$ with the sign changes in the above sum make all the terms cancel by pairs.

<u>**References**</u>

R. Charney, M. Davis, The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold, *Pacific J. Math.* 171 (1995), no. 1, 117–137.

N. C. Leung, V. Reiner, The signature of a toric variety, *math.AG/0111064*.

Notation

(M, ω)	generic notation for a symplectic manifold
$\Omega^k(M, \mathbb{R})$	space of (real) k-forms on M
T_pM	tangent space of a point p of M
$\mathcal{X}(M)$	vector fields on M
S^k	k-dimensional sphere
S^1	1-dimensional sphere (circle), and group of rotations in $\mathbb{C}$
ξ_M	vector field induced by an action of a torus T on M
$\mathcal{L}$	Lie derivative
$\imath_{\xi_M}$	map defined by $\imath_{\xi_M}\omega(a) = \omega(\xi_M, a)$
ϕ	moment map associated to an action of a torus T on (M, ω)
ϕ^ξ	component of ϕ in the ξ direction: $\phi^\xi(x) = \langle \phi(x), \xi \rangle$
$H^k(M, \mathbb{R})$	de Rham cohomology groups
$[\sigma]$	cohomology class of σ
T^k	k-dimensional torus $(S^1)^k$
$\mathrm{Stab}\, y$	stabilizer of y
M^T	fixed points of M under an action of a torus T
$M /\!\!/ S^1$	reduced space of (M, ω) under an action of S^1
$\mathbb{C}\,\mathbb{P}^n$	complex n-dimensional projective space
$SU(n)$	Lie group of determinant 1 unitary $n \times n$ matrices
$\mathfrak{su}(n)$	Lie algebra of $SU(n)$
$\mathrm{Symp}(M, \omega)$	groups of symplectomorphisms $(M, \omega) \longrightarrow (M, \omega)$
$\mathfrak{t}, \mathfrak{t}^*$	Lie algebra of a torus T and its dual
$\mathfrak{l}$	lattice in $\mathfrak{t}$
$\mathrm{SL}(n, \mathbb{Z})$	group of determinant 1 $n \times n$ matrices with integer coefficients
Δ	(Delzant) polytope
M_Δ	toric variety associated to a Delzant polytope Δ
$H^*(M)$	cohomology ring of M
$c_n(M)$	nth Chern class of M
$\beta_i(M)$	ith Betti number of M
$h(\Delta)$	h-vector of Δ