LECTURE 3: MORSE THEORY AND EQUIVARIANT COHOMOLOGY

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We will assume from now on that the symplectic manifold (M, ω) is compact and connected, and that S^1 acts on M with moment map ϕ.

Recall that $x \in M$ is a critical point of *phi* if and only if it is fixed by the action.

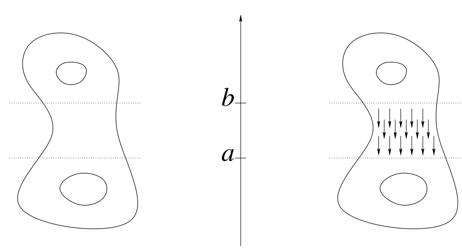
The goal of this lecture is understanding $H^*(M)$, the cohomology of M.

1 Morse theory

Lemma (Morse)

If (a, b) contains no critical values, then $\phi((-\infty, a))$ is homotopy equivalent to $\phi((-\infty, b))$ (denoted $\phi((-\infty, a)) \sim \phi((-\infty, b))$.

Further, for a fixed metric g on M, there is a unique vector field $\nabla\phi \in \mathcal{X}(M)$ such that $g(\nabla\phi, X) = X(\phi)\ \forall X \in \mathcal{X}(M)$.



Note The lemma is not specific to moment maps; it is actually true for any function $M \longrightarrow \mathbb{R}$.

Recall that locally, M is symplectomorphic to $\mathbb{C}^n$, and the moment map of the action of S^1 given by $\lambda \cdot z = (\lambda^{\eta_1} z_1, \ldots, \lambda^{\eta_n} z_n)$ takes the form $\phi(z) = \sum \eta_i |z_i|^2$. If $p \in M^{S^1}$ is an isolated fixed point, then the η_i are nonzero near p. The η_i are called *weights.*

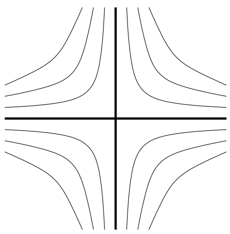
Definition *Say that ϕ is a* Morse function *if it can be written in the form $\phi(z) = \sum \eta_i |z_i|^2$ near its critical points (fixed points of the action).*

If p is an isolated fixed point, define the index λ_p *of p to be twice the number of negative weights (twice the number of negative η_i).*

More generally, a function f on a manifold is Morse if locally it can be written as a sum of quadratics in the coordinates: $f = \sum \eta_i |z_i|^2$. While the weights η_i depend on the choice of coordinates, the number of negative ones is an invariant, so we can still define the index as twice the number of negative weights. In the case of a symplectic manifold with a circle action preserved by f, the η_i are actually well-defined up to permutation.

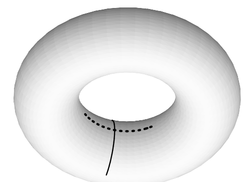
We will assume from now on that the fixed points are isolated.

Toy picture



$\phi(x, y) = |x|^2 - |y|^2$

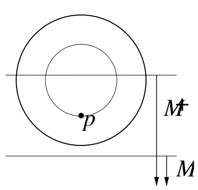
(saddle point)

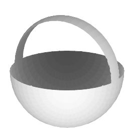


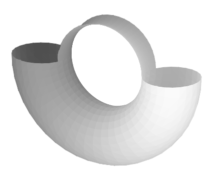
Theorem *Let $M^{\pm} = \phi^{-1}((-\infty, \phi(p) \pm \varepsilon))$ for ε sufficiently small. If D^λ is the unit disk in $\mathbb{R}^\lambda$ and $S^{\lambda-1}$ the sphere of dimension $\lambda - 1$, then*

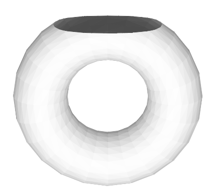
$$M^+ \sim M^- \cup_{S^{\lambda-1}} D^\lambda \,,$$

where $M^- \cup_{S^{\lambda-1}} D^\lambda$ is the result of glueing to M^- the disk D^λ along its boundary $S^{\lambda-1}$ (the glueing map is not specified here).









Corollary *From general principles, we have the long exact sequence*

$$\cdots \longrightarrow H^*(M^+, M^-) \longrightarrow H^*(M^+) \longrightarrow H^*(M^-) \longrightarrow \cdots$$

With the previous theorem, we can write

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^*(M^+, M^-) & \longrightarrow & H^*(M^+) & \longrightarrow & H^*(M^-) & \longrightarrow \cdots \\
& \wr\| & & & & & \\
& H^*(D^\lambda, S^{\lambda-1}) & \cong & \tilde{H}^*(S^\lambda) & & & \\
& \wr\| & & & & & \\
& H^{*-\lambda}(\text{point}) & & & & & \\
& \| & & & & & \\
& 0 & \text{unless } * = \lambda & & & &
\end{array}$$

($H^{*-\lambda}$ means we shift down the exponent by λ.)

More generally, suppose $F \subseteq M^{S^1}$ is a connected component, and that λ is the number of negative weights. If $N(F)$ is the negative normal bundle of F, then there exists a λ-dimensional bundle $\begin{array}{ccc} E & \subseteq & N(F) \\ \downarrow & & \\ F & & \end{array}$ such that

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^*(M^+, M^-) & \longrightarrow & H^*(M^+) & \longrightarrow & H^*(M^-) & \longrightarrow & \cdots \\ H^*(D(E), S(E)) & \cong & \cong & & \downarrow & & & & \\ & \cong & H^{*-\lambda}(F) & \xrightarrow{\times e} & H^*(F) & & & & \end{array}$$

where $\times e$ denotes multiplication by e, the Euler class of E, and $D(E)$, $S(E)$ are the disk and sphere bundles of E.

2 Equivariant cohomology

Suppose a group G acts on M (think of a torus action). There always exists a space EG (not necessarily a manifold) which is contractible and on which G acts freely.

Definition *The* equivariant cohomology $H^*_G(M)$ *is defined to be* $H^*(M \times_G EG)$.

Example *If* $G = S^1$ *then* $EG = S^\infty$ *and*

$$H^*_{S^1}(\text{point}) = H^*\left(S^\infty / S^1\right) = H^*(\mathbb{C}\mathbb{P}^\infty) = \mathbb{C}[X^2]\,.$$

Note If G acts freely on M, then $H^*_G(M) = H^*\left(M/G\right)$.

Line bundles

If G acts on a bundle $\begin{array}{c} E \\ \downarrow \\ F \end{array}$ and fixes F, we get

$$\begin{array}{ccccc} \tilde{E} & = & E \times_G EG & & \\ & & \downarrow & & \\ & & F \times_G EG & = & F \times EG/G \end{array}$$

to classify equivariant bundles. Denote by $\tilde{e}(\tilde{E})$ the Euler class of $\tilde{E}$.

Example *Let* S^1 *act on the bundle* $\begin{array}{c} \mathbb{C}^n \\ \downarrow \\ \bullet \text{ (point)} \end{array}$ *with weights* η_i.

Then $\tilde{e}(\widetilde{\mathbb{C}^n}) = \left(\prod \eta_i\right) X^n$ *(same* X *as in the example of equivariant cohomology above). In particular, if* $\eta_i \neq 0\ \forall i$*, this is not a zero divisor.*

Theorem *Let $F \subseteq M^{S^1}$ be a connected component, and λ be twice the number of negative weights.*

There is a λ-dimensional bundle $\begin{smallmatrix} E \\ \downarrow \\ F \end{smallmatrix}$ such that

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\boxed{1}} & H^*_{S^1}(M^+, M^-) & \xrightarrow{\boxed{2}} & H^*_{S^1}(M^+) & \xrightarrow{\boxed{3}} & H^*_{S^1}(M^-) & \xrightarrow{\boxed{4}} & \cdots \\
 & & \wr\| & & \downarrow & & & & \\
 & & H^{*-\lambda}_{S^1}(F) & \xrightarrow{\times \tilde{e}} & H^*_{S^1}(F) & & & & \\
 & & \wr\| & & & & & & \\
 & & H^*(F) \otimes H^*_{S^1}(\mathrm{pt}) & & & & & &
\end{array}$$

<u>**Claim**</u> **(Atiyah-Bott)**

$\tilde{e}$ has no zero divisors, i.e. $\tilde{e} \cdot z \neq 0$ if $z \neq 0$.

Corollary *In the diagram above, $\times\tilde{e}$ is injective, and thus $\boxed{1} = \boxed{4} = 0$, $\boxed{2}$ is injective and $\boxed{3}$ surjective.*

Corollary *The restriction $H^*_{S^1}(M) \longrightarrow H^*_{S^1}(M^{S^1})$ is one-to-one.*

Corollary *$H^*_{S^1}(M) \simeq H^*(M) \otimes H^*_{S^1}(\mathrm{pt})$ (as vector spaces, not as rings). In fact, if we let $BG = EG/G$, then $H^*_{S^1}(M) \simeq H^*(M) \otimes H^*(BG)$ (not as rings).*

Corollary

$$0 \longrightarrow H^*(M^+, M^-) \longrightarrow H^*(M^+) \longrightarrow H^*(M^-) \longrightarrow 0$$

is exact. Also note (recall) that $H^*(M^+, M^-) \cong H^{*-\lambda}(F)$ and $H^*(M) \cong H^*_{S^1}(M)/H^*_{S^1}(\mathrm{pt})$ (as rings).

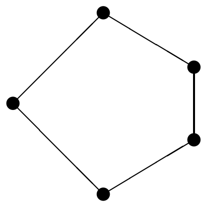
Definition *The Poincaré polynomials of a space X is defined as*

$$P(X) = \sum \dim H^i(X)\, t^i\,.$$

Corollary *$P(M) = \sum_F t^{\lambda_F} P(F)$.*

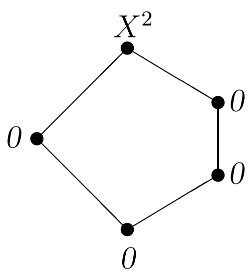
Corollary *Assume all the fixed points are isolated. Then for every $p \in M^{S^1}$ there is a (almost unique) $\alpha_p \in H^*_{S^1}(M)$ such that $\alpha_{p|_p} = \tilde{e}(E)$ and $\alpha_{p|_{p'}} = 0$ for all p' with $\phi(p') < \phi(p)$. Furthermore, these α_p form a vector space basis for $H^*_{S^1}(M)$.*

Example

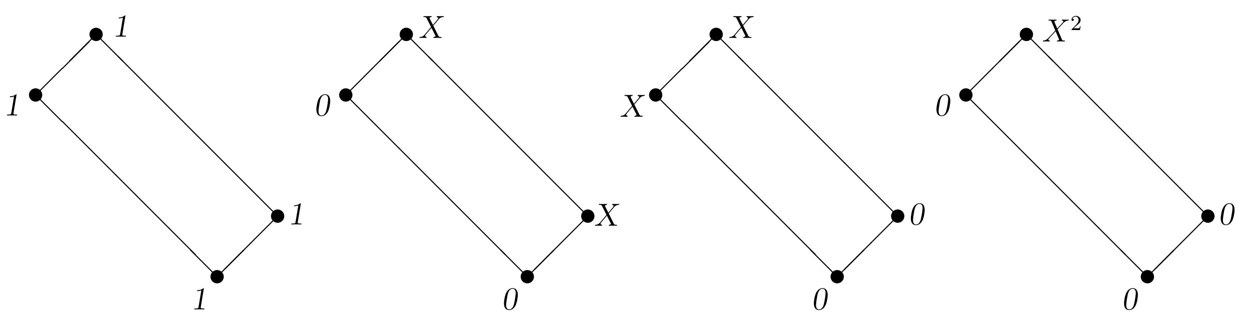


index 2
Euler class = -X $\Rightarrow \exists$ *class whose restriction to this point is* $-X$

index 0 $\Rightarrow \exists$ *class whose restriction to this point is 1*

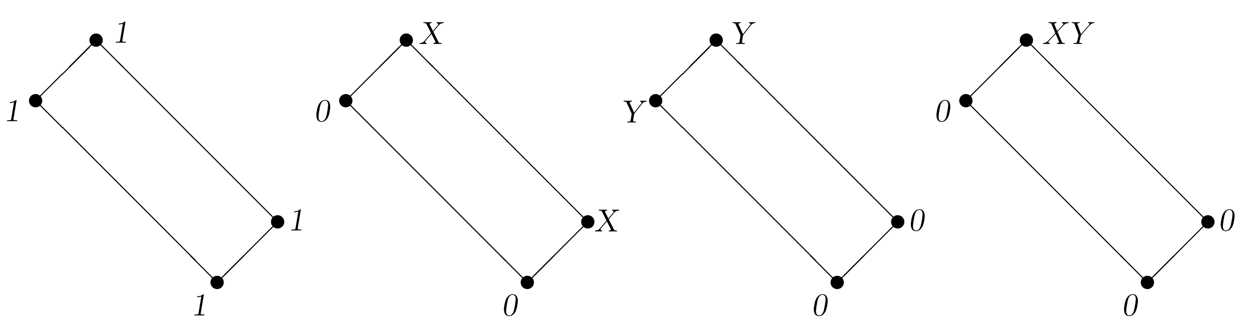


Example $S^2 \times S^2$



Note This carries over to T-actions with moment map $\phi : M \longrightarrow \mathfrak{t}^*$ by fixing $\xi \in \mathfrak{t}$ and considering ϕ^ξ.

Example *T^2 acts on $S^2 \times S^2$*



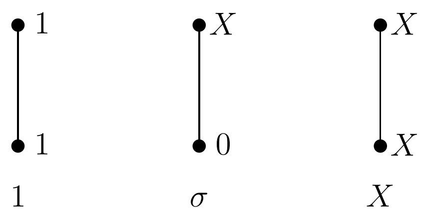
Discussion

$$\begin{array}{c} M \\ \downarrow \\ p \text{ (point)} \end{array} \quad \text{induces} \quad \begin{array}{c} M \times_G EG \\ \downarrow \\ p \times_G EG \end{array} \quad \text{which in turn induces } H_G^*(p) \longrightarrow H_G^*(M).$$

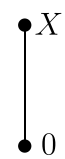
In our case, this map is one-to-one.

Example

$H^*_{S^1}(S^2) = \mathbb{C}[X,\sigma] / \langle \sigma(X-\sigma) \rangle$. *The generators are*

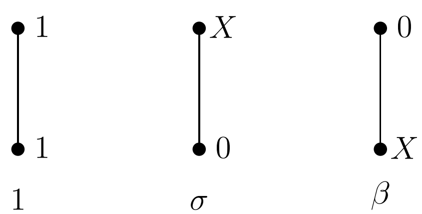


So $X - \sigma$ *would be*



So $H^*_{S^1}(S^2) / H^*_{S^1}(\mathrm{pt}) = \mathbb{C}[\sigma] / \langle \sigma^2 \rangle$ *(set* $X = 0$*).*

This is compatible with the construction of the previous lecture, where we got $\mathbb{C}[x_1, x_2] / \langle a - b, ab \rangle$. *In the context of that lecture, the generators would be*



Notation

(M,ω)	generic notation for a symplectic manifold
$\Omega^k(M,\mathbb{R})$	space of (real) k-forms on M
T_pM	tangent space of a point p of M
$\mathcal{X}(M)$	vector fields on M
S^k	k-dimensional sphere
S^1	1-dimensional sphere (circle), and group of rotations in $\mathbb{C}$
ξ_M	vector field induced by an action of a torus T on M
$\mathcal{L}$	Lie derivative
$\imath_{\xi_M}$	map defined by $\imath_{\xi_M}\omega(a) = \omega(\xi_M, a)$
ϕ	moment map associated to an action of a torus T on (M,ω)
ϕ^ξ	component of ϕ in the ξ direction: $\phi^\xi(x) = \langle \phi(x), \xi \rangle$
$H^k(M,\mathbb{R})$	de Rham cohomology groups
$[\sigma]$	cohomology class of σ
T^k	k-dimensional torus $(S^1)^k$
$\mathrm{Stab}\, y$	stabilizer of y
M^T	fixed points of M under an action of a torus T
$M /\!\!/ S^1$	reduced space of (M,ω) under an action of S^1
$\mathbb{C}\mathbb{P}^n$	complex n-dimensional projective space
$SU(n)$	Lie group of determinant 1 unitary $n \times n$ matrices
$\mathfrak{su}(n)$	Lie algebra of $SU(n)$
$\mathrm{Symp}(M,\omega)$	groups of symplectomorphisms $(M,\omega) \longrightarrow (M,\omega)$
$\mathfrak{t}, \mathfrak{t}^*$	Lie algebra of a torus T and its dual
$\mathfrak{l}$	lattice in $\mathfrak{t}$
$\mathrm{SL}(n,\mathbb{Z})$	group of determinant 1 $n \times n$ matrices with integer coefficients
Δ	(Delzant) polytope
M_Δ	toric variety associated to a Delzant polytope Δ
$H^*(M)$	cohomology ring of M
$c_n(M)$	nth Chern class of M
$\beta_i(M)$	ith Betti number of M
$h(\Delta)$	h-vector of Δ
η_i	weights of a moment map
λ_p, λ_F	index of an isolated fixed point p or a fixed component F
D^λ	disk of dimension λ
$N(F)$	negative normal bundle
$D(E)$, $S(E)$	disk and sphere bundles of E
e	Euler class of E
EG	classifying space
$H^*_G(M)$	equivariant cohomology of M
$P(X)$	Poincaré polynomial