**LECTURE 4: THE DUISTERMAAT-HECKMAN MEASURE**

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# 1 Definition of the D-H measure

Let $(M^{2n}, \omega)$ be a symplectic manifold.

**Definition** *A* Borel set *in $M$ is a set generated from compact subsets of $M$ under countable union and complementation.*

**Definition** *Given a Borel set $U$ in $M$, the* Liouville measure *of $U$ is defined as*

$$\mathrm{vol}(U) = \int_U \frac{\omega^n}{(2\pi)^n n!}\,.$$

Let a torus $T$ act on $(M, \omega)$ with *proper* moment map $\phi : M \longrightarrow \mathfrak{t}^*$. (A moment map $\phi$ is *proper* if $\phi^{-1}(K)$ is compact whenever $K$ is.)

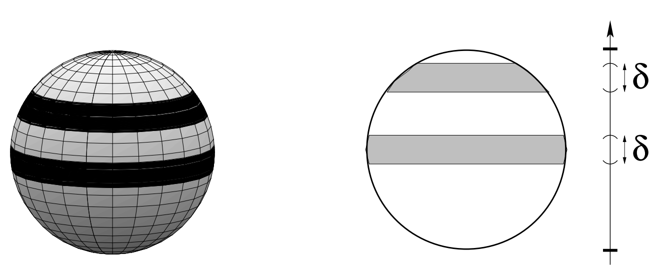
**Definition** *The* Duistermaat-Heckman measure $m = m_{DH}$ *on $\mathfrak{t}^*$ is the push-forward of the Liouville measure. Thus, for $U \subseteq \mathfrak{t}^*$ Borel,*

$$m(U) = \mathrm{vol}(\phi^{-1}(U)) = \int_{\phi^{-1}(U)} \frac{\omega^n}{(2\pi)^n n!}\,.$$

It follows from the definition that the support of the D-H measure lies inside the image of $M$ under $\phi$ : $\mathrm{supp}(m) \subseteq \phi(M)$, because $U \cap \phi(M) = \emptyset \ \Rightarrow\ m(U) = 0$.

**Theorem (Archimedes, ∼230 BC)**

*The area on the sphere between two latitudes depends only on the difference of their heights along the rotation axis.*



**Proof** Let $S^1$ act on $S^2$ by rotation ($S^2$ embedded in $\mathbb{R}^3$ in the usual way). If the rotation is around the $z$-axis, $\phi(x, y, z) = z$. The image of $S^2$ under $\phi$ is the interval $[-1, 1]$.

The definition of the D-H measure gives that for $[a, b] \subseteq [-1, 1]$, $m([a, b]) = b - a$. ■

## 2 The D-H polynomial

**Theorem (Duistermaat-Heckman)**

*There is a function $f : \mathfrak{t}^* \longrightarrow \mathbb{R}$ such that*

1. *$f$ is a polynomial of degree at most $\frac{1}{2}\dim M - \dim T$ on each component of regular values of $\phi$;*
2. *$m(U) = \int_U f \, d\lambda$ ($\lambda$ is the Lebesgue measure).*

**Note** The D-H measure is absolutely continuous with respect to the Lebesgue measure.

**Example** *In the example above ($S^1$ acting on $S^2$), we expect $f$ to be of degree at most $\frac{1}{2}\dim S^2 - \dim S^1 = 0$, i.e. a constant on the connected component of regular values* $(-1, 1)$, *and indeed $f$ is the characteristic function $\chi_{[-1,1]}$ of the interval* $[-1, 1]$.

**Note** $f$ is called the *Duistermaat-Heckman polynomial*, even though it is really piecewise polynomial.

**Fact** Whenever $\frac{1}{2}\dim M = \dim T$, $f$ will not only be a constant, but actually be either 0 or 1.

**Example** *$S^1$ acts on $\mathbb{C}$ by $\lambda \cdot z = \lambda z$, and the moment map of this action is $\phi(z) = \frac{1}{2}|z|^2$ (see first lecture). The image of $\mathbb{C}$ under $\phi$ is $\mathbb{R}_{\geq 0}$. Computing the D-H measure from the definition, we get*

$$m([0, b]) = \frac{1}{2\pi}(\text{area of the disk of radius } \sqrt{2b}\,) = \frac{1}{2\pi} 2\pi b = b\,,$$

*so that for $[a, b] \subseteq \mathbb{R}_{\geq 0}$, $m([a, b]) = b - a$.*

*Thus the D-H polynomial is $\chi_{\mathbb{R}_{\geq 0}}$.*

**Example** *$(S^1)^n$ acts on $\mathbb{C}^n$ by $\lambda \cdot z = (\lambda_1 z_1, \ldots, \lambda_n z_n)$. The image of $\mathbb{C}^n$ under the moment map $\phi(z) = \frac{1}{2}\sum |z_i|^2$ is $(\mathbb{R}_{\geq 0})^n$. Then $(\mathbb{R}_{>0})^n$ is a connected component of regular values and the D-H polynomial is $\chi_{(\mathbb{R}_{\geq 0})^n}$.*

**Example** *Since $\frac{1}{2}\dim M = \dim T$ for toric varieties, the D-H polynomial on any toric variety $(M, \omega, \phi)$ will be $\chi_{\phi(M)}$.*

## 3 Behavior of the D-H measure under projections

Let $T$ act on $M$ with moment map $\phi : M \longrightarrow \mathfrak{t}^*$. Given a subgroup $H$ of $T$, we get the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{t}$ and a projection $p : \mathfrak{t}^* \longrightarrow \mathfrak{h}^*$.

**Fact** The moment map $\psi : M \longrightarrow \mathfrak{h}^*$ for the $H$-action is $\psi = p \circ \phi$.

So for $U \subseteq \mathfrak{h}^*$,

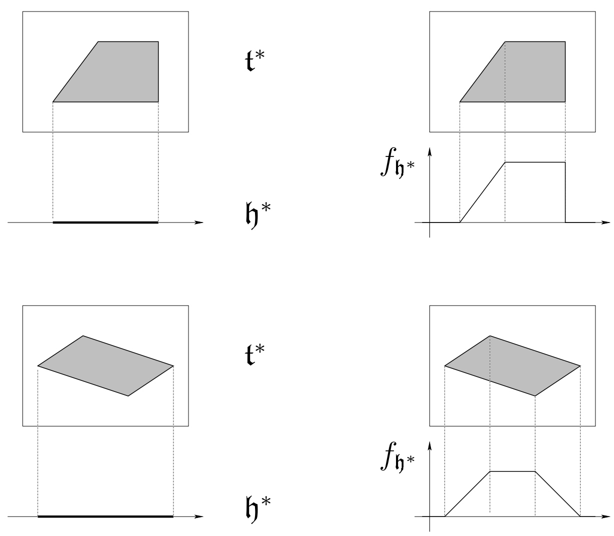
$$m_{\mathfrak{h}^*}(U) = \mathrm{vol}\big(\psi^{-1}(U)\big) = \mathrm{vol}\big(\phi^{-1}(p^{-1}(U))\big) = m_{\mathfrak{t}^*}(p^{-1}(U))\,.$$

($m_{\mathfrak{h}^*}$ is called the *push-forward measure*.)

The D-H polynomial also behaves nicely : for $a \in \mathfrak{h}^*$,

$$f_{\mathfrak{h}^*}(a) = \int_{p^{-1}(a)} f_{\mathfrak{t}^*}(p^{-1}(a))\, d\lambda\,.$$

**Example** $f_{\mathfrak{h}^*}$ *is the "thickness" of the fiber above a projected point.*

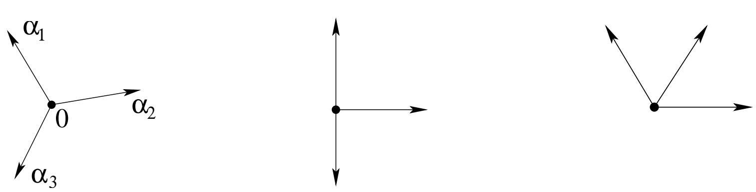



Let $T < (S^1)^n$ act on $\mathbb{C}^n$ via $\lambda \cdot z = (\lambda_1^{\alpha_1} z_1, \ldots, \lambda_n^{\alpha_n} z_n)$. The projection $p : (\mathbb{R}^n)^* \longrightarrow \mathfrak{t}^*$ sends the standard basis element $e_i$ to $\alpha_i$.

The moment map $\psi : \mathbb{C}^n \longrightarrow \mathfrak{t}^*$ is given by $\psi(z) = \frac{1}{2} \sum \alpha_i |z_i|^2$. Therefore

$$\Delta = \psi(\mathbb{C}^n) = \{s_1\alpha_1 + \cdots + s_n\alpha_n \mid s_1, \ldots, s_n \geq 0\}\,.$$

We also find that $\psi$ is proper if and only if $\Delta$ is properly contained in a half-space (or equivalently, does not contain a line).

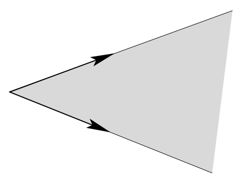



not proper  not proper  proper

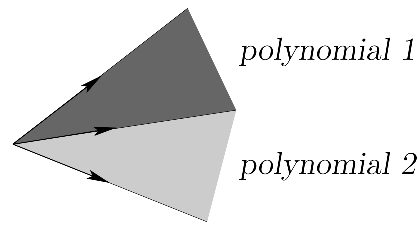
For $\psi$, the D-H function on $\mathfrak{t}^*$ is given by

$$f(a) = \mathrm{vol}\{s_1, \ldots, s_n \geq 0 \mid a = s_1\alpha_1 + \cdots + s_n\alpha_n\}\,.$$

**Example**



*$S^1$ on $\mathbb{C}^2$*




*Projection from $\mathbb{R}^3$ to $\mathbb{R}^2$*

# 4 Computing the D-H measure

A way to compute the D-H measure comes out of the proof of the Duistermaat-Heckman theorem, so we give an idea of the proof here.

For the sake of simplicity, suppose that $T = S^1$. Assume that 0 is a regular value of the moment map, and that $t \in \mathbb{R}$ is near 0.

Let $(M_t, \omega_t)$ denote the reduced space at $t$. If we let $Z = \phi^{-1}(0)$, then we have the bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & Z \\ & & \big\downarrow \pi \\ & & M_0 \end{array}$$

Let $\alpha$ be a connection one-form, i.e. find $\alpha$ such that $\imath_{\xi_M}\alpha = 1$ and $\mathcal{L}_{\xi_M}\alpha = 0$; then $d\alpha$ is basic (the pull-back of a form $\beta$ on $M$). So $d\alpha = \pi^*(\beta)$ ($\beta$ is the curvature and is in the cohomology class of $c_1$).

**<u>Fact</u>** Near 0, $M \approx Z \times (-\varepsilon, \varepsilon)$, $\phi(z, t) = t$ and $\omega \approx \pi^*(\omega_0) - d(\alpha t)$.

So $M_t \approx M_0$ and $\omega_t = \omega_0 - t\beta$ and thus the symplectic form varies linearly. So

$$\begin{aligned} \mathrm{vol}(M_t) &= \int_{M_0} ([\omega_0] - t[\beta])^{n-1}) \qquad \left(n - 1 = \frac{1}{2}\dim M - \dim S^1\right) \\ &= \sum \binom{n-1}{k} \left(\int_{M_0} [\omega_0]^k [\beta]^{n-1-k}\right) t^k\,. \end{aligned}$$

$[\omega]$ and $[\beta]$ are constant cohomology classes (don't depend on $t$). Therefore $\mathrm{vol}(M_t)$ is a polynomial in $t$. So it is straightforward to compute the D-H function:

$$f(t) = \mathrm{vol}\big((M_t, \omega_t)\big)\,.$$

# 5 Computing the D-H polynomial combinatorially

We will assume from this point on that $M$ is compact and that the set of fixed points $M^T$ is finite.

For each $p \in M^T$, let the weights at $p$ be $\alpha_p^1, \ldots, \alpha_p^n \in \mathfrak{t}^*$. Pick $\xi \in \mathfrak{t}$ such that the inner product $(\alpha_p^i, \xi)$ is never zero.

For each $p$, define $\beta_p^i \in \mathfrak{t}^*$ by

$$\beta_p^i = \begin{cases} \alpha_p^i & \text{if } (\alpha_p^i, \xi) > 0\,, \\ -\alpha_p^i & \text{if } (\alpha_p^i, \xi) < 0\,. \end{cases}$$

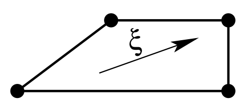
Also let $w_p$ be the number of $\alpha_p^i$ with $(\alpha_p^i, \xi) < 0$.

**Definition** *For $a \in \mathfrak{t}^*$, let*

$$f_p(a + \phi(p)) = \mathrm{vol}\{s_1, \ldots, s_n \mid s_1\beta_p^1 + \cdots + s_n\beta_p^n = a\}\,.$$

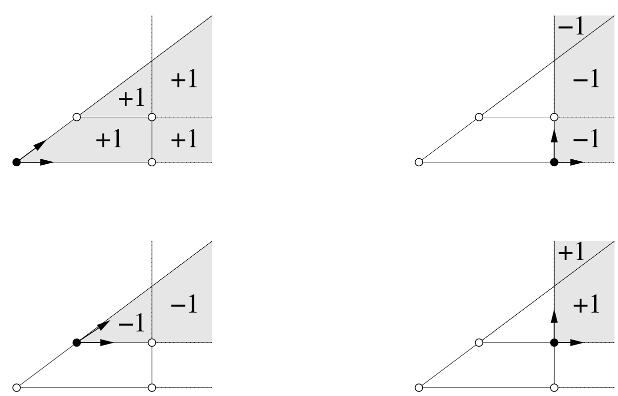
**Theorem (Guillemin-Lerman-Sternberg, after Atiyah-Bott)**

*The D-H polynomial is* $\sum_p (-1)^{w_p} f_p$.

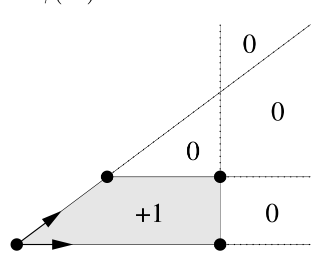
**Example** *Consider* 

*with 4 fixed points and the privileged direction* $\xi$ *as indicated.*

*For each vertex $p$ (fixed point), we compute* $(-1)^{w_p} f_p$*:*




*Adding up gives the D-H function* $\chi_{\phi(M)}$

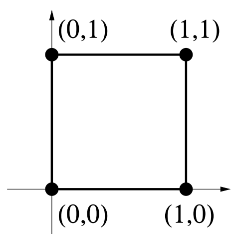


# Discussion

## Lattice points inside a polytope

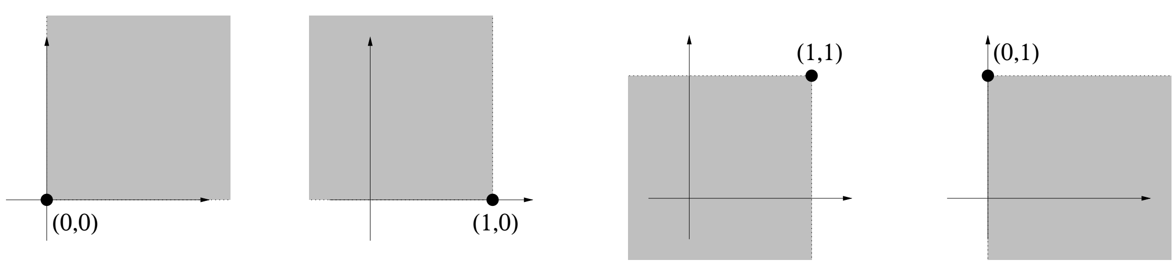
Something similar to the computation of the D-H function using the Guillemin-Lerman-Sternberg formula occurs when counting integer lattice points inside a polytope (with integer vertices), using the monomial weight $x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}$ for the lattice point $(i_1, i_2, \ldots, i_n)$ (if we are in $\mathbb{Z}^n$). For each vertex on the polytope, we consider the cone at that vertex pointing inside the polytope. The total weight of the lattice points inside that cone is a rational function of the $x_i$. If we add up all the weights of the vertex cones, we get the weight of the integer points in the polytope.

For example consider the square




$$x^0y^0 + x^1y^0 + x^0y^1 + x^1y^1 = 1 + x + y + xy$$

The cones and their weights are




$$\frac{1}{(1-x)(1-y)} = \frac{1}{(1-x)(1-y)}$$

$$\frac{x}{(1-x^{-1})(1-y)} = \frac{-x^2}{(1-x)(1-y)}$$

$$\frac{xy}{(1-x^{-1})(1-y^{-1})} = \frac{x^2y^2}{(1-x)(1-y)}$$

$$\frac{y}{(1-x)(1-y^{-1})} = \frac{-y^2}{(1-x)(1-y)}$$

And they sum up to

$$\frac{1-x^2-y^2+x^2y^2}{(1-x)(1-y)} = \frac{(1-x^2)(1-y^2)}{(1-x)(1-y)} = (1+x)(1+y) = 1+x+y+xy\,.$$

## Weight multiplicities

Let $\mathfrak{g}$ be a semisimple Lie algebra and fix a root system. If $\lambda$ is a dominant weight, then there is a (unique up to isomorphism) irreducible $\mathfrak{g}$-module $V(\lambda)$ with highest weight $\lambda$. For $\mu$ in the weight lattice, we can ask what the dimension of the weight space $V(\lambda)_\mu$ is in the weight space decomposition of $V(\lambda)$. This dimension is called the multiplicity of $\mu$ in the representation $V(\lambda)$.

To get weight multiplicities instead of the D-H measure (which is a sort of limiting case), the volumes have to be replaced by the numbers of integer lattice points inside the corresponding polytopes, and $a \in \mathfrak{t}^*$ has to be replaced by $a + \rho$ ($\rho$ is half the sum of the positive roots).

## Stationary phase formula

Suppose we have a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that 0 is a critical point ($\frac{\partial f}{\partial x_i} = 0 \ \forall i$) and such that the Hessian $\left(\frac{\partial^2}{\partial x_i \partial x_j}\right)_{i,j}$ at 0 is non-degenerate.

Then for $t \gg 0$ and $g$ compactly supported near 0, we get the asymptotic formula

$$\int e^{itf} g \ dx_1 \dots dx_n \sim \frac{it^{n/2}}{\sqrt{\det\left(\left(\frac{\partial^2}{\partial x_i \partial x_j}\right)_{i,j}\right)}} .$$

If $M$ is symplectic and compact, $f$ the $S^1$-moment map, and $dx_1 \dots dx_n = \omega^n / n!$, then the formula above is exact (no asymptotics).

# References


[1] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics 1764, Springer, 2001.

[2] V. Guillemin, E. Lerman, S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge University Press, 1996.

# Notation

| | |
|---|---|
| $(M, \omega)$ | generic notation for a symplectic manifold |
| $\Omega^k(M, \mathbb{R})$ | space of (real) $k$-forms on $M$ |
| $T_pM$ | tangent space of a point $p$ of $M$ |
| $\mathcal{X}(M)$ | vector fields on $M$ |
| $S^k$ | $k$-dimensional sphere |
| $S^1$ | 1-dimensional sphere (circle), and group of rotations in $\mathbb{C}$ |
| $\xi_M$ | vector field induced by an action of a torus $T$ on $M$ |
| $\mathcal{L}$ | Lie derivative |
| $\imath_{\xi_M}$ | map defined by $\imath_{\xi_M}\omega(a) = \omega(\xi_M, a)$ |
| $\phi$ | moment map associated to an action of a torus $T$ on $(M, \omega)$ |
| $\phi^\xi$ | component of $\phi$ in the $\xi$ direction: $\phi^\xi(x) = \langle \phi(x), \xi \rangle$ |
| $H^k(M, \mathbb{R})$ | de Rham cohomology groups |
| $[\sigma]$ | cohomology class of $\sigma$ |
| $T^k$ | $k$-dimensional torus $(S^1)^k$ |
| $\mathrm{Stab}\, y$ | stabilizer of $y$ |
| $M^T$ | fixed points of $M$ under an action of a torus $T$ |
| $M /\!\!/ S^1$ | reduced space of $(M, \omega)$ under an action of $S^1$ |
| $\mathbb{C}\mathbb{P}^n$ | complex $n$-dimensional projective space |
| $SU(n)$ | Lie group of determinant 1 unitary $n \times n$ matrices |
| $\mathfrak{su}(n)$ | Lie algebra of $SU(n)$ |
| $\mathrm{Symp}(M, \omega)$ | groups of symplectomorphisms $(M, \omega) \longrightarrow (M, \omega)$ |
| $\mathfrak{t}, \mathfrak{t}^*$ | Lie algebra of a torus $T$ and its dual |
| $\mathfrak{l}$ | lattice in $\mathfrak{t}$ |
| $\mathrm{SL}(n, \mathbb{Z})$ | group of determinant 1 $n \times n$ matrices with integer coefficients |
| $\Delta$ | (Delzant) polytope |
| $M_\Delta$ | toric variety associated to a Delzant polytope $\Delta$ |
| $H^*(M)$ | cohomology ring of $M$ |
| $c_n(M)$ | $n$th Chern class of $M$ |
| $\beta_i(M)$ | $i$th Betti number of $M$ |
| $h(\Delta)$ | $h$-vector of $\Delta$ |
| $\eta_i$ | weights of a moment map |
| $\lambda_p$, $\lambda_F$ | index of an isolated fixed point $p$ or a fixed component $F$ |
| $D^\lambda$ | disk of dimension $\lambda$ |
| $N(F)$ | negative normal bundle |
| $D(E)$, $S(E)$ | disk and sphere bundles of $E$ |
| $e$ | Euler class of $E$ |
| $EG$ | classifying space |
| $H^*_G(M)$ | equivariant cohomology of $M$ |
| $P(X)$ | Poincaré polynomial |
| vol | Liouville measure |
| $m, m_{DH}$ | Duistermaat-Heckman measure |
| $\chi_X$ | characteristic function of set $X$ ($\chi_X(a) = 1$ if $a \in X$ and 0 otherwise) |
| $f, f_{\mathfrak{t}^*}, f_{\mathfrak{h}^*}$ | Duistermaat-Heckman polynomial (function) |