COHOMOLOGY OF A HAMILTONIAN T-SPACE WITH INVOLUTION

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Abstract. Let M be a compact symplectic manifold on which a compact torus T acts Hamiltonialy with a moment map μ. Suppose there exists a symplectic involution $\theta : M \to M$, such that $\mu \circ \theta = -\mu$. Assuming that 0 is a regular value of μ, we calculate the character of the action of θ on the cohomology of M in terms of the character of the action of θ on the symplectic reduction $\mu^{-1}(0)/T$ of M. This result generalizes a theorem of R. Stanley, who considered the case when M was a toric variety and $\dim T = \frac{1}{2}\dim_{\mathbb{R}} M$.

1. Introduction

In [7], R. Stanley proved a lower bound for the number of faces of a centrally symmetric simple polytope. The main ingredient of his proof is the following result: Suppose M is a toric variety corresponding to a simple centrally symmetric polytope P of dimension n. Then M is acted upon by an n-dimensional torus T. The symmetry on P induces an involution $\theta : M \to M$, such that

$$\theta \circ t \ = \ t^{-1} \circ \theta, \qquad \text{for any} \quad t \in T. \tag{1.1}$$

Then θ acts on the cohomology $H^{2i}(M, \mathbb{C})$ of M. The result of Stanley states that the character of this action is equal to $\binom{n}{i}$.

In this paper, we extend the result of Stanley to an arbitrary compact symplectic manifold M, which possesses a Hamiltonian action of a torus T of arbitrary dimension and an involution θ, which preserves the symplectic form and satisfies (1.1). In this situation, the moment map for the action of T can be chosen so that $\mu \circ \theta = -\theta$. Assuming that 0 is a regular value of μ, we compute (cf. Theorem 2.2) the character of the action of θ on the cohomology $H^*(M, \mathbb{C})$ in terms of the action of θ on the symplectic reduction $M_0 = \mu^{-1}(0)/T$ of M. Note that, if $\dim T = \frac{1}{2}\dim M$, so that M is a toric variety, then M_0 is a point and our formula reduces to the theorem of Stanley. Hence, we obtain, in particular, a new, more geometric proof, of Stanley's result.

The proof of our main theorem is based on a study of the action of θ on the equivariant cohomology of M. More precisely, we compute the *graded character* $\chi_\theta(t)$ of this action (cf. (2.4)) in two different ways.

First, we uses the well known formula $H^*_T(M) = H^*(M) \otimes H_T(pt)$ to express $\chi_\theta(t)$ in terms of the character of the action of θ on $H^*(M)$, cf. Proposition 2.6.

Our second computation (cf. Proposition 2.7) expresses $\chi_\theta(t)$ in terms of the character of the action of θ on the cohomology of the symplectic reduction $H^*(M_0)$. This is done in the following way: We consider the square $f = \langle \mu, \mu \rangle$ of the moment map as a Morse-Bott function on M. It is equivariant with respect to the action of the semi-direct product $\mathbb{Z}_2 \ltimes T = \widetilde{T}$ on M (here the action of $\mathbb{Z}_2$ is generated by θ). In Lemma 4.3 we show that the $\widetilde{T}$-equivariant Morse inequalities for f *with local coefficients* are, in fact, equalities and, hence, may be used to calculate the equivariant cohomology of M. This generalizes a result of Kirwan, [6, Th. 5.4]. Comparing the above Morse *equalities* with different local coefficients, we calculate $\chi_\theta(t)$ via the character of the action of θ on M_0.

Comparison of the above two expressions for $\chi_\theta(t)$ leads to a proof of our main theorem.

Contents. The paper is organized as follows:

In Section 2 we formulate and prove our main result – Theorem 2.2. The proof is based on two statements (Propositions 2.6 and 2.7), which are proven in the later sections.

In Section 3 we present some examples and applications of Theorem 2.2. In particular, we show that it implies the result of Stanley [7]. We also discuss applications of our theorem to flag varieties.

In Section 4 we study the action of the involution θ on the equivariant cohomology of M and prove Proposition 2.6 (the expression of $\chi_\theta(t)$ in terms of the action of θ on $H^*(M)$).

Finally, in Section 5, we use the equivariant Morse inequalities to prove Proposition 2.7 (the expression of $\chi_\theta(t)$ in terms of the action of θ on $H^*(M_0)$).

2. Main theorem

2.1. Let (M,ω) be a compact $2n$-dimensional symplectic manifold endowed with a Hamiltonian action of a compact k-dimensional torus T. Let $\mathfrak{t}^*\simeq\mathbb{R}^k$ denote the dual space to the Lie algebra of T and let $\mu : M\to\mathfrak{t}^*$ be the moment map for the action of T on M.

Let $\theta : M\to M$ be an involution of M such that $\theta^*\omega=\omega$ and

$$\theta(t\cdot m) \;=\; (t^{-1})\cdot m, \quad \text{for any} \quad t\in T,\ m\in M. \tag{2.1}$$

We can and we will normalize the moment map (which is defined up to an additive constant) so that

$$\mu\circ\theta = -\mu. \tag{2.2}$$

In this situation θ acts naturally on the cohomology $H^i(M)=H^i(M;\mathbb{C})$ of M with complex coefficients. Let $H^i(M)^+$ (resp. $H^i(M)^-$) denote the subspace of $H^i(M)$ fixed by θ (resp. the subspace of $H^i(M)$ on which θ acts as a multiplication by -1). Set

$$h^{i,\pm} = \dim_{\mathbb{C}} H^i(M)^{\pm}.$$

Suppose now that zero is a regular value for the moment map μ. Then $\mu^{-1}(0)$ is a smooth manifold which is invariant under the actions of T and θ. Moreover, the action of T on $\mu^{-1}(0)$ is *locally free*, i.e. each point of $\mu^{-1}(0)$ has at most finite stabilizer in T. Hence, the symplectic reduction $M_0 := \mu^{-1}(0)/T$ is an *orbifold*. Let $H^i(M_0)$ denote the cohomology of M_0 with complex coefficients.

The involution θ preserves $\mu^{-1}(0)$ and, hence, acts on M_0 and $H^i(M_0)$. Let $h_0^{i,+}$ (resp. $h_0^{i,-}$) denote the dimension of the subspace of θ invariant vectors in $H^i(M_0)$ (resp. the dimension of the subspace of the vectors on which θ acts by multiplication by -1).

Our principal result is the following

Theorem 2.2. *In the situation described above*

$$h^{i,+} \;-\; h^{i,-} \;=\; \sum_{j=0}^{\min(k,[i/2])} \binom{k}{j}\,(h_0^{i-2j,+}-h_0^{i-2j,-}), \tag{2.3}$$

for any $i=0,1,\dots,2n$.

Remark 2.3. a. The conditions of the theorem imply that θ acts freely on the set C of fixed points of any subtorus T' of T. Moreover, θ acts freely on the set of connected components of C. It follows from the fact that, if μ' is the moment map for the T'-action, then $\mu' \circ \theta = -\theta \circ \mu'$ and the restriction of μ' on any connected component is a non-zero constant.

b. The theorem remains true if M is a symplectic orbifold, rather than a smooth manifold. The proof is just a bit more complicated than the one we present is this paper.

c. Moreover, the theorem may be generalized to the case when M has more serious singularities (say, to the case when M is a singular algebraic manifold). In this case, the usual cohomology must be replaced by the intersection cohomology. The use of the intersection cohomology also allows to release the assumption that 0 is a regular value of the moment map. The details will appear elsewhere.

d. The special case of the theorem, when M is a toric manifold, is due to R. Stanley [7]. In this sense, our result is a generalization of Stanley's theorem. In particular, we obtain a new, more geometric proof, of the Stanley's theorem. See Subsection 3.1 for details.

2.4. The proof of the theorem is based on a study of the action of θ on the equivariant cohomology of M. We now formulate the main results about this action. The proofs are given on Sections 4 and 5. Some examples and applications of Theorem 2.2 are discussed in Section 3.

2.5. **Action of the involution on the equivariant cohomology.** Let ET denote the universal T-bundle and let $BT = ET/T$ denote the classifying space of T. Let $H^*_T(M) = H^*(ET \times_T M; \mathbb{C})$ denote the equivariant cohomology of M with complex coefficients. Then θ acts naturally on $H^*_T(M)$, cf. Subsection 4.1. Let

$$\chi_\theta(t) \ := \ \sum_{t=0}^{\infty} t^i \operatorname{Tr} \theta|_{H^i_T(M)} \tag{2.4}$$

denote the *graded character* of this action.

In Section 4 we use the fact that the spectral sequence of the fibration $M \otimes_T ET \to BT$ degenerates at the second term (cf. [6, Proof of Pr. 5.8], [4, Proof of Th. 5.3]) to prove the following proposition, in which we do not assume that 0 is a regular value of the moment map.

Proposition 2.6. $\chi_\theta(t) \ = \ \sum_{i=1}^{2n}(h^{i,+} - h^{i,-})\dfrac{t^i}{(1+t^2)^k}.$

On the other side, in Section 5 we use a version of the equivariant Morse inequalities [1] constructed in [3, 4] to get the following

Proposition 2.7. *Suppose that zero is a regular value for the moment map μ and let $M_0 := \mu^{-1}(0)/T$. Then χ_θ is equal to the graded character of the action of θ on $H^*(M_0)$:*

$$\chi_\theta(t) \ = \ \sum_i (h_0^{i,+} - h_0^{i,-})t^i. \tag{2.5}$$

Corollary 2.8. *If, in the conditions of Proposition 2.7, the dimension of the torus is equal to $n = \frac{1}{2}\dim_{\mathbb{R}} M$, then $\chi_\theta = 1$.*

2.9. **Proof of the main theorem.** Comparing Propositions 2.6 and 2.7 we obtain

$$\sum_{i=1}^{2n}(h^{i,+}-h^{i,-})\frac{t^i}{(1+t^2)^k} \;=\; \sum_i (h_0^{i,+}-h_0^{i,-})t^i,$$

which is, obviously, equivalent to Theorem 2.2. □

3. Examples and applications

3.1. **Symmetric toric variety. Application to combinatorics.** Theorem 2.2 takes a particularly simple form if the dimension of the torus is equal to $n=\frac{1}{2}\dim_{\mathbb{R}} M$, so that M is a toric variety. In this case we will say that M is a *symmetric (with respect to the involution θ) toric variety.*

If M is a symmetric toric variety, then the reduced space M_0 is a point. Hence, $h_0^{i,-}=0$ for all i, $h_0^{i,+}=0$ for all $i>0$ and $h_0^{0,+}=1$. The Theorem 2.2 reduces in this case to the following statement, which was originally proven by R. Stanley by a completely different method [1].

Corollary 3.2. *If, in the conditions of Theorem 2.2, the dimension of the torus is equal to $n=\frac{1}{2}\dim_{\mathbb{R}} M$, then*

$$h^{i,+} \;-\; h^{i,-} \;=\; \binom{n}{i}.$$

There are a lot of examples of symmetric toric varieties. To describe these examples let us recall that each toric variety is completely characterized by its *moment polytope* $\mu(M)\subset\mathfrak{t}^*$. The toric variety M is an orbifold if and only if the polytope $\mu(M)$ is *simple*, i.e. if each of its vertices has the valence $n=\frac{1}{2}\dim_{\mathbb{R}} M$. There is also a complete description of polytopes corresponding to smooth toric varieties, cf., for example, [2, §IV.2]. Now, the equation (2.2) implies that the moment polytope corresponding to a symmetric toric variety is centrally symmetric. Vice versa, if the moment polytope is centrally symmetric, one easily constructs an involution on M satisfying (2.2). Hence, symmetric toric orbifolds are in one-to-one correspondence with centrally symmetric simple convex polytopes. Corollary 3.2 can be used to get an estimate on the number of faces of such a polytope. See [7] for details.

Remark 3.3. Recently, A. A´Campo-Neuen [5] extended Corollary 3.2 to singular toric varieties. This leads to an extension of Stanley's estimates on the number of faces of a centrally symmetric polytopes to rational polytopes, which are not necessarily simple. This result may be also obtained by our method, cf. Remark 2.3.c.

3.4. $\mathbb{C}P^3$ **as an S^1-space with involution.** Some toric varieties, which are not symmetric, still posses an involution compatible, in the sense of (2.1), with the action of a torus of smaller dimension. Before discussing more general examples, let us consider the simplest case $M=\mathbb{C}P^3$. Then the moment polytope is a 3-dimensional simplex, which is, obviously, not centrally symmetric. Hence, there is no involution on M compatible, in the sense of (2.1), with the action of a 3-dimensional torus. However, using the homogeneous coordinates $[z_1:z_2:z_3:z_4]$ on M, we can define the action of the circle $T=S^1$ on M and an involution $\theta: M\to M$ by

$$t\cdot[z_1:z_2:z_3:z_4] \;=\; [tz_1:tz_2:z_3:z_4], \qquad t\in S^1=\{\zeta\in\mathbb{C}:\,|\zeta|=1\}$$
$$\theta\cdot[z_1:z_2:z_3:z_4] \;=\; [z_3:z_4:z_1:z_2].$$

Clearly, all the conditions of Theorem 2.2 are satisfied.

[1]Stanley proved the theorem for more general case, when M is a symmetric toric orbifold. One can prove this result using our method and Remark 2.3.b

In this case, both sides of (2.3) may be easily calculated. In particular, $M_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ and θ acts on M_0 by $\theta : (a, b) \mapsto (b, a)$, where $a, b \in \mathbb{C}P^1$. We leave to the interested reader to verify Theorem 2.2 in this simple case.

3.5. **Toric varieties with involution.** The previous example allows the following generalization. Let M be a toric variety endowed with an action of the torus T of dimension $n = \frac{1}{2}\dim_{\mathbb{R}} M$. Suppose that there exists a linear involution $\widetilde{\theta} : \mathfrak{t}^* \to \mathfrak{t}^*$ which preserves the moment polytope $\mu(M)$. Then $\widetilde{\theta}$ induces an involution $\theta : M \to M$. However, in general, this involution is not compatible with the action of T in the sense of (2.1).

Let $\widetilde{\theta}^* : \mathfrak{t} \to \mathfrak{t}$ denote the involution of the Lie algebra $\mathfrak{t}$ dual to $\widetilde{\theta}$. Let $T' \subset T$ be a subtorus, such that $\widetilde{\theta}^*$ acts as multiplication by -1 on the Lie algebra of T'. Then the actions of T' and θ on M are compatible in the sense of (2.1). If, in addition, 0 is a regular value of the moment map for the T' action, then Theorem 2.2 may be applied.

3.6. **The variety of complete flags in $\mathbb{C}^n$.** We now give an example of a manifold, which is not toric, but satisfies the conditions of Theorem 2.2.

Let $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a centrally symmetric set of real numbers. In other words, $\lambda = \{\pm\nu_1, \ldots, \pm\nu_k\}$ if $n = 2k$ and $\lambda = \{0, \pm\nu_1, \ldots, \pm\nu_k\}$ if $n = 2k+1$. We will assume that the numbers ν_i above are positive and mutually different. The set A_λ of all complex Hermitian $n \times n$-matrices with spectrum λ is naturally identified with the variety of complete flags in $\mathbb{C}^n$. In particular, A_λ has a structure of a compact Kähler manifold.

Fix mutually different rational numbers $r_1, \ldots, r_n$ and consider the action of the circle $S^1 = \{e^{it} : t \in \mathbb{R}\}$ on A_λ, defined by

$$e^{it} \cdot A \;=\; \operatorname{diag}\left\{ e^{ir_1 t}, \ldots, e^{ir_n t} \right\} \;\cdot\; A \;\cdot\; \operatorname{diag}\left\{ e^{-ir_1 t}, \ldots, e^{-ir_n t} \right\}, \qquad t \in \mathbb{R},\; A \in A_\lambda. \tag{3.1}$$

This action is Hamiltonian with respect to the Kähler structure on A_λ: if we identify the coalgebra Lie of S^1 with $\mathbb{R}$, then the function

$$\mu :\, A_\lambda \;\to \mathbb{R}, \qquad A = \{a_{ij}\} \;\mapsto\; \sum_{i=1}^{n} r_i a_{ii}$$

is a moment map for this action.

Define an involution $\theta : A_\lambda \to A_\lambda$ by the formula

$$\theta :\, A \;\mapsto\; -A^t.$$

This involution preserves the Kähler structure on A_λ and satisfies (2.1), (2.2). Moreover, one easily checks that S^1 acts locally freely on $\mu^{-1}(0)$. Hence, zero is a regular value of μ and Theorem 2.2 is applicable.

Remark 3.7. a. The action (3.1) is a restriction of the conjugate action on A_λ of the torus of all unitary diagonal matrices. The latter action is also Hamiltonian and compatible with the involution θ in the sense of (2.1). Unfortunately, zero is not a regular value of the moment map for this action. However, the generalization of Theorem 2.2, indicated in Remark 2.3.c, may be applied in this case.

b. One easily generalizes the results of this subsection to a flag variety of an arbitrary reductive group G. The involution θ should be replaced by the action of the longest element of the Weyl group. Also the action of the maximal torus of G should be restricted to a subtorus on whose Lie algebra the longest element of the Weyl group acts as a multiplication by -1, cf. Subsection 3.5. We leave the details to the interested reader.

4. Proof of Proposition 2.6

4.1. **Action of θ on $H^*_T(M)$.** Before proving the proposition let us describe more explicitly the action of θ on $H^*_T(M)$. One of the ways to define this action is the following. Consider the group

$$\widetilde{T} \;:=\; \big\{\,(\varepsilon,t):\,\varepsilon=\pm1, t\in T\,\big\}$$

with the product law

$$(\varepsilon_1,t_1)\cdot(\varepsilon_2,t_2) \;=\; (\varepsilon_1\varepsilon_2, t_1^{\varepsilon_2}t_2).$$

Then $\widetilde{T}$ acts on M by the formula

$$(\varepsilon,t)\cdot m \;=\; t^{\varepsilon}\cdot\theta^{\varepsilon}(m).$$

We will identify θ with the element $(-1,1)\in\widetilde{T}$. Let $E\widetilde{T}$ denote the universal $\widetilde{T}$-bundle. Then θ acts diagonally on $E\widetilde{T}\times_T M$ and, hence, on $H^*_T(M)=H^*(E\widetilde{T}\times_T M)$ (in the last equality we used that $E\widetilde{T}$ may be considered also as a model for the universal T-bundle).

4.2. **Action of the involution on BT.** We consider the quotient $BT=E\widetilde{T}/T$ as a model for the classifying space of T. Then $\theta=(-1,1)\in\widetilde{T}$ acts on BT and, hence, on the cohomology ring $H^*(BT)$. The later is isomorphic to the graded ring $\mathbb{C}[\mathfrak{t}]$ of polynomials on the Lie algebra $\mathfrak{t}$ of T (with grading given by twice the degree of the polynomial):

$$H^*(BT) \;\simeq \mathbb{C}[\mathfrak{t}]. \tag{4.1}$$

The following lemma describes the action of θ on this ring.

Lemma 4.3. *Under the isomorphism* (4.1) *the action of θ on $H^*(BT)$ is given by the rule*

$$\theta(t_i) \;=\; -t_i.$$

In other words, the restriction of θ on $H^2(BT)$ is equal to -1.

Proof. This lemma is well known, but in view of the difficulty we have to locate an explicit reference, we present a proof here. For simplicity, we only present the proof for the case $\dim_{\mathbb{R}} T=1$. The arguments in general case are exactly the same, but the notation is more complicated. We also identify the one dimensional torus with the unit circle $S^1=\{e^{i\phi}:\,\phi\in\mathbb{R}\}$ in $\mathbb{C}$.

Fix $m\ge3$ and consider the free action of $\widetilde{T}$ on the product of spheres

$$S^{2m-1}\times S^{2m-1} \;:=\; \big\{\,(z_1,\dots,z_m;w_1,\dots,w_m)\in\mathbb{C}^{2m}:\,|z_1|^2+\cdots|z_m|^2=1;\,|w_1|^2+\cdots|w_m|^2=1\big\},$$

given by

$$\begin{aligned} e^{i\phi}\cdot(z_1,\dots,z_m;w_1,\dots,w_m) \;&\mapsto (e^{i\phi}z_1,\dots,e^{i\phi}z_m;w_1,\dots,w_m);\\ (-1,0)\cdot(z_1,\dots,z_m;w_1,\dots,w_m) \;&\mapsto (\overline{z}_1,\dots,\overline{z}_m;-w_1,\dots,-w_m). \end{aligned}$$

Let us denote by BT_m the quotient of $S^{2m-1}\times S^{2m-1}$ by the action of $T\subset\widetilde{T}$. Since the reduced cohomology of $S^{2m-1}\times S^{2m-1}$ vanishes up to dimension $2m-2\ge4$ it follows that there is a natural $\widetilde{T}/T$ equivariant isomorphism between the cohomology $H^i(BT_m)$ and $H^i(BT)$ for any i less than $2m-3\ge3$ (this may be shown by the same arguments as in the proof of Lemma 2.8 in [4])

Clearly, θ acts on $BT_m=\mathbb{CP}^{m-1}\times S^{2m-1}$ as complex conjugation on the first factor and multiplication by -1 on the second factor. Hence, the induced action on the cohomology $H^2(BT_m)$ is multiplication by -1. This proves the lemma. $\square$

Corollary 4.4. *The graded character of the action of θ on $H^*(BT)$ is equal to $(1+t^2)^k$.*

4.5. **The spectral sequence. Proof of Proposition 2.7.** The projection $E\widetilde{T}\times M\to E\widetilde{T}$ induces a $\widetilde{T}/T$ equivariant fiber bundle $p: E\widetilde{T}\times_T M\to BT$. The fiber of this bundle is $\widetilde{T}/T$ equivariantly homeomorphic to M. The cohomology $H^*(E\widetilde{T}\times_T M)=H^*_T(M)$ may be calculated by the spectral sequence of the above bundle. The later spectral sequence degenerates at the second term (cf. [6, Proof of Pr. 5.8], [4, Proof of Th. 5.3]) and is obviously $\widetilde{T}/T$ invariant. It follows that there exists a $\widetilde{T}/T$ equivariant isomorphism of graded rings

$$H^*_T(M)\ \simeq\ H^*(M)\otimes_{\mathbb{C}} H^*(BT).$$

Hence, the graded character $\chi_\theta(t)$ of the action of θ on $H^*_T(M)$ is equal to the product of the graded characters of the actions of θ on $H^*(M)$ and $H^*(BT)$. Proposition 2.7 follows now from Corollary 4.4. □

5. Equivariant Morse inequalities and proof of Proposition 2.7

Our proof of Proposition 2.7 is an application of the $\widetilde{T}$-equivariant Morse inequalities for the square of the moment map. Since the group $\widetilde{T}$ is disconnected, it is important to use the equivariant Morse inequalities with local coefficients, cf. [3, 4].

The key fact of the proof is that the square of the moment map is an *equivariantly perfect Morse function*, i.e. the above inequalities are, in fact, equalities, cf. Lemma 5.5. This result is a slight generalization of a theorem of Kirwan [6, 6, Th. 5.4], who considered T-equivariant Morse inequalities with trivial local coefficients.

5.1. **Equivariant Morse inequalities.** For convenience of the reader we recall here equivariant Morse inequalities with local coefficients for an action of a disconnected group, as they formulated in [3, 4]. In this subsection M is a compact manifold acted upon by a compact, not necessarily connected, Lie group G. Let $\mathcal{F}$ be a G-equivariant flat vector bundle over M. Denote by $H^*_G(M,\mathcal{F})$ the G-equivariant cohomology of M with coefficients in $\mathcal{F}$ and let

$$\mathcal{P}^G_{\mathcal{F}}(t)\ =\ \sum_i t^i\dim_{\mathbb{C}} H^i_G(M,\mathcal{F})$$

be the *equivariant Poincaré series* of M with coefficients on $\mathcal{F}$.

Suppose $f: M\to\mathbb{R}$ is a G-invariant function on M and let C denote the set of critical points of f. We assume that f is *non-degenerate in the sense of Bott*, i.e. C is a submanifold of M and the Hessian of f is a non-degenerate on the normal bundle $\nu(C)$ to C in M.

Let Z be a connected component of the critical point set C and let $\nu(Z)$ denote the normal bundle to Z in M. The bundle $\nu(Z)$ splits into the Whitney sum of two subbundles $\nu(Z)=\nu^+(Z)\oplus\nu^-(Z)$, such that the Hessian is strictly positive on $\nu^+(Z)$ and strictly negative on $\nu^-(Z)$. The dimension of the bundle $\nu^-(Z)$ is called the *index* of Z (as a critical submanifold of f) and is denoted by $\mathrm{ind}(Z)$. Let $o(Z)$ denote the *orientation bundle of $\nu^-(Z)$, considered as a flat line bundle*.

If the group G is connected, then Z is a G-invariant submanifold of M. In general, we denote by $G_Z=\{g\in G|\ g\cdot Z\subset Z\}$ the stabilizer of the component Z in G. Let $|G:G_Z|$ denote the index of G_Z as a subgroup of G; it is always finite.

The compact Lie group G_Z acts on the manifold Z and the flat vector bundles $\mathcal{F}|_Z$ and $o(Z)$ are G_Z-equivariant. Let $H^*_{G_Z}(Z,\mathcal{F}|_Z\otimes o(Z))$ denote the equivariant cohomology of the flat G_Z-equivariant vector bundle $\mathcal{F}|_Z \otimes o(Z)$. Consider the *equivariant Poincaré series*

$$\mathcal{P}^{G_Z}_{Z,\mathcal{F}}(t) \;=\; \sum_i t^i \dim_{\mathbb{C}} H^i_{G_Z}(Z, \mathcal{F}|_Z \otimes o(Z))$$

and define, using it, the following *equivariant Morse counting series*

$$\mathcal{M}^G_{f,\mathcal{F}}(t) \;=\; \sum_Z t^{\operatorname{ind}(Z)} |G : G_Z|^{-1} \mathcal{P}^{G_Z}_{Z,\mathcal{F}}(t)$$

where the sum is taken over all connected components Z of C.

The following version of the equivariant Morse inequalities is a particular case of [3, Th. 7],[4, Th. 1.7].

Theorem 5.2. *Suppose that G is a compact Lie group, acting on a closed manifold M, and let $\mathcal{F}$ be an equivariant flat vector bundle over M. Then for any non-degenerate (in the sense of Bott) G-equivariant function $f : M \to \mathbb{R}$, there exists a formal power series $\mathcal{Q}(t)$ with non-negative integer coefficients, such that*

$$\mathcal{M}^G_{f,\mathcal{F}}(t) \;-\; \mathcal{P}^G_{\mathcal{F}}(t) \;=\; (1+t)\mathcal{Q}(t).$$

5.3. **Equivariant flat bundles.** We now return to the situation described in Section 2. Let $\mathcal{F}$ be a $\widetilde{T}$-equivariant flat vector bundle over M. This is the same as T-equivariant flat vector bundle, on which θ acts preserving the connection and so that $\theta(t \cdot \xi) = (t^{-1}) \cdot \xi$ for any $\xi \in \mathcal{F}, t \in T$.

In our proof of Proposition 2.7 we will only use the following type of $\widetilde{T}$-equivariant bundles: Let $\rho : \mathbb{Z}_2 \to \operatorname{End}_{\mathbb{C}}(V_\rho)$ be a representation of $\mathbb{Z}_2$ in a finite-dimensional complex vector space V_ρ. Consider the bundle $\mathcal{F}_\rho = M \times V_\rho$ with the trivial connection and with the action of $\widetilde{T}$ given by

$$(\varepsilon, t) : \, (m, \xi) \mapsto \big((\varepsilon, t) \cdot m, \, \rho(\varepsilon) \cdot \xi \big).$$

Then the the equivariant cohomology $H^*_{\widetilde{T}}(M, \mathcal{F}_\rho)$ of M with coefficients in $\mathcal{F}_\rho$ is given by

$$H^*_{\widetilde{T}}(M, \mathcal{F}_\rho) \;=\; \operatorname{Hom}_{\mathbb{Z}_2}\big(V^*_\rho, H^*_T(M)\big) \tag{5.1}$$

(here V^*_ρ is the representation dual to V_ρ). In particular, if ρ is the regular representation of $\mathbb{Z}_2$ then

$$H^*_{\widetilde{T}}(M, \mathcal{F}_\rho) = H^*_T(M). \tag{5.2}$$

The bundle $\mathcal{F}_\rho$ is completely determined by the representation ρ. When it causes no confusion, we will write ρ for $\mathcal{F}_\rho$ in oder to simplify the notation.

5.4. **Morse equalities for the square of the moment map.** Fix a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{t}^*$ and consider the real-valued function $f = \langle \mu, \mu \rangle$ on M. Then f is a $\widetilde{T}$ invariant Morse function on M. Moreover, if Z is any connected component of the set C of critical points of f, then the dimension of Z is even and the bundles $\nu^{\pm}(Z)$ (cf. Subsection 5.1) are orientable. In particularly, the orientation bundle $o(Z)$ is trivial.

The set $Z_0 := \mu^{-1}(0)$ is a connected component of the set of critical points of f. The group $\widetilde{T}$ acts on Z_0 and the equivariant cohomology

$$H^*_{\widetilde{T}}(Z_0, \mathcal{F}_\rho|_{Z_0}) \;=\; \operatorname{Hom}_{\mathbb{Z}_2}(V^*_\rho, H^*(M_0)).$$

The equivariant Poincaré series of Z_0 with coefficients in $\mathcal{F}_\rho$ is given by

$$\mathcal{P}^{\widetilde{T}}_{Z_0,\rho}(t) \;=\; \sum_i t^i \dim_{\mathbb{C}} H^*_{\widetilde{T}}(Z_0, \mathcal{F}_\rho|_{Z_0}) \;=\; \sum_i t^i \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{Z}_2}(V^*_\rho, H^*(M_0)).$$

In particular,

$$\begin{aligned} \mathcal{P}^{\widetilde{T}}_{Z_0,\rho}(t) &= \sum_i t^i h_0^{i,+}, \quad \text{if } \rho \text{ is the trivial representation;} \\ \mathcal{P}^{\widetilde{T}}_{Z_0,\rho}(t) &= \sum_i t^i h_0^{i,-}, \quad \text{if } \rho \text{ is the sign representation.} \end{aligned} \tag{5.3}$$

The stabilizer G_{Z_0} of Z_0 coincides with the whole group $\widetilde{T}$. Thus $|G : G_{Z_0}| = 1$.

The involution θ acts freely on the set of the connected components of C different from Z_0, cf. Remark 2.3.a. Hence, if $Z \neq Z_0$ is a connected component of C, then $G_Z = T$ and $|G : G_Z| = 2$. The T-equivariant Poincaré polynomial of Z is independent of ρ and is given by

$$\mathcal{P}^T_Z(t) \;=\; \dim_{\mathbb{C}} V_\rho \cdot \sum_i t^i \dim_{\mathbb{C}} H^i_T(Z, \mathcal{F}|_Z).$$

Hence, the equivariant Morse counting series

$$\mathcal{M}^{\widetilde{T}}_{f,\rho(t)} \;=\; \mathcal{P}^{\widetilde{T}}_{Z_0,\rho}(t) \;+\; \frac{1}{2}\sum_Z \mathcal{P}^T_Z(t).$$

Here the sum in the right hand side is taken over all connected components of the set of critical points of f different from Z_0.

Let

$$\mathcal{P}^{\widetilde{T}}_\rho(t) \;=\; \sum_i t^i \dim_{\mathbb{C}} H^i_{\widetilde{T}}(M, \mathcal{F}_\rho)$$

be the equivariant Poincaré series of M with coefficients in $\mathcal{F}_\rho$.

The following lemma expresses the fact that f is an *equivariantly perfect Morse function*.

Lemma 5.5. *The following equality holds*

$$\mathcal{M}^{\widetilde{T}}_{f,\rho}(t) \;=\; \mathcal{P}^{\widetilde{T}}_\rho(t). \tag{5.4}$$

Proof. It follows from Theorem 5.2, that there exists a formal power series $Q_\rho(t)$ with non-negative coefficients, such that

$$\mathcal{M}^{\widetilde{T}}_{f,\rho(t)} \;=\; \mathcal{P}^{\widetilde{T}}_\rho(t) \;=\; \sum_i t^i H^i_{\widetilde{T}}(M, \mathcal{F}_\rho) \;+\; (1+t)Q_\rho(t). \tag{5.5}$$

Our goal is to show that $Q_\rho \equiv 0$.

It follows from (5.1), that both the equivariant Morse counting series and the equivariant Poincaré series are additive with respect to ρ. More precisely, if $\rho_1 \oplus \rho_2$ denotes the direct sum of two representations then

$$\mathcal{M}^{\widetilde{T}}_{f,\rho_1\oplus\rho_2}(t) \;=\; \mathcal{M}^{\widetilde{T}}_{f,\rho_1}(t) \;+\; \mathcal{M}^{f,\widetilde{T}}_{\rho_2}(t), \quad \mathcal{P}^{\widetilde{T}}_{\rho_1\oplus\rho_2}(t) \;=\; \mathcal{P}^{\widetilde{T}}_{\rho_1}(t) \;+\; \mathcal{P}^{\widetilde{T}}_{\rho_2}(t).$$

Hence, it suffices to prove the lemma for the irreducible representations of $\mathbb{Z}_2$. Moreover, it follows from (5.5) that it is enough to prove that $Q_\rho = 0$ when ρ is a reducible representation, which contains any of the irreducible representation as a subrepresentation.

In particular, it is enough to prove that $Q_\rho = 0$ when ρ is the regular representation. However, (5.2) implies that, if ρ is the regular representation, then (5.5) reduces to the T-equivariant Morse inequalities with trivial coefficients. It was shown by Kirwan [6, Th. 5.4] that the later inequalities are exact, i.e. $Q_\rho = 0$. $\square$

5.6. **Proof of Proposition 2.7.** Set

$$H_T^*(M)^{\pm} \ := \ \big\{ x \in H_T^*(M) : \theta x = \pm x \big\}$$

Let us, first, apply Lemma 5.5 with ρ being the trivial representation of $\mathbb{Z}_2$. It follows from (5.1) that, in this case, $\mathcal{P}_\rho^{\widetilde{T}}(t) = \sum_i t^i \dim_{\mathbb{C}} H_T^i(M)^+$. Hence, from (5.3) and Lemma 5.5 we obtain

$$\sum_i t^i \dim_{\mathbb{C}} H_T^i(M)^+ \ = \ \frac{1}{2} \sum_Z \mathcal{P}_Z^T(t) \ + \ \sum_i t^i h_0^{i,+}. \tag{5.6}$$

Apply now Lemma 5.5 with ρ being the sign representation of $\mathbb{Z}_2$. Then $\mathcal{P}_\rho^{\widetilde{T}}(t) = \sum_i t^i \dim_{\mathbb{C}} H_T^i(M)^-$ and $\mathcal{P}_{Z_0,\rho}^{\widetilde{T}} = \sum_i t^i h_0^{i,-}$. Hence,

$$\sum_i t^i \dim_{\mathbb{C}} H_T^i(M)^- \ = \ \frac{1}{2} \sum_Z \mathcal{P}_Z^T(t) \ + \ \sum_i t^i h_0^{i,-}. \tag{5.7}$$

Subtracting (5.7) from (5.6) we get (2.5). $\square$

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