MY FAVORITE OPEN PROBLEMS

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This note offers a collection of open problems, mostly about Brownian motion. I tried to solve every problem on the list but I failed to do so. Although I do not recall seeing these problems published anywhere, I do not insist on being their author—other probabilists might have proposed them independently.

1. Probabilistic version of McMillan's Theorem in higher dimensions. Let X_t be the d-dimensional Brownian motion starting from the origin, and let D be an open set in the d-dimensional space containing the origin. Let $\tau = \inf\{t > 0 : X_t \notin D\}$ be the exit time from D. Consider the set A of "asymptotic directions of approach," depending on the domain D and the trajectory of X_t, and defined as the set of all cluster points of

$$\frac{X_t - X_\tau}{|X_t - X_\tau|},$$

as $t \uparrow \tau$. It has been proved in Burdzy (1990a) that for $d = 2$, a.s., the set A is equal to either a circle or a semicircle. In some domains D, the set A is a circle with a non-trivial probability, i.e., with a probability strictly between 0 and 1.

Problem 1. *Is it true that for every $d > 2$ and every d-dimensional open set D, the set A of asymptotic directions of approach is either a sphere or a hemisphere, a.s.?*

2. Topology of planar Brownian trace. Let X_t be the two-dimensional Brownian motion. A Jordan arc is a set homeomorphic to a line segment.

Problem 2 (i) . *Is it true that for every pair of points $x, y \notin X[0,1]$ one can find a Jordan arc Γ with $x, y \in \Gamma$, and such that $\Gamma \cap X[0,1]$ contains a finite number of points, possibly depending on x and y?*

Let $\{A_k\}_{k\geq 1}$ be the family of all connected components of the complement of $X[0,1]$, and let $K = X[0,1] \setminus \bigcup_{k\geq 1} \partial A_k$. We say that a set is totally disconnected if it has no connected subsets containing more than one point.

Problem 2 (ii). *Is K totally disconnected?*

The negative answer to Problem 2 (ii) and a soft argument would yield the negative answer to Problem 2 (i). It is easy to see that for any fixed $t \in [0,1]$, $X_t \in K$, a.s. Hence, the dimension of K is equal to 2, a.s. The problem is related to the existence of "cut points;" see Burdzy (1989, 1995). It is also related to the question of whether $X[0,1]$ is a "universal planar curve" or equivalently, whether it contains a homeomorphic image of the Sierpiński carpet; see Mandelbrot (1982, Section VIII.25).

3. Percolation dimension of planar Brownian trace. Let $\dim(A)$ denote the Hausdorff dimension of a set A. The *percolation dimension* of a set B is the infimum of $\dim A$, where the infimum is taken over all Jordan arcs $A \subset B$ which contain at least two distinct points. Suppose that X_t is a two-dimensional Brownian motion.

Problem 3. *Is the percolation dimension of $X[0,1]$ equal to 1?*

Related paper: Burdzy (1990b).

4. Efficient couplings in acute triangles. Suppose that D is a triangle with all angles strictly smaller than $\pi/2$ and let $\mu_2 >$ be the second eiganvalue for the Laplacian in D with Neumann boundary conditions. The first eigenvalue is zero.

Problem 4. *Can one construct two reflected Brownian motions X_t and Y_t in D starting from different points and such that $\tau = \inf\{t \geq 0 : X_t = Y_t\} < \infty$ a.s., and for every fixed $\varepsilon > 0$,*

$$P(\tau > t) \leq \exp(-(\mu_2 - \varepsilon)t),$$

for large t?

See Burdzy and Kendall (2000) for the background of the problem.

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