Math 407, Winter 2006

Definitions : Chapters 1–3

Chapter 1

- *Linear Function:* A linear function on $\mathbb{R}^n$ is any function of the form
$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n,$$
where $a = [a, \dots, a_n]^T \in \mathbb{R}^n$.

- *Linear Inequality:* A linear inequality is an inequality that can be written in one of the following two forms:
$$a^T x \le b \qquad \text{or} \qquad a^T x \ge b,$$
where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

- *The solution set of a system of linear inequalities:* Given $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, we write $Ax \le b$ to denote the system of linear inequalities
$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1, \dots, m.$$
The set of solutions of this system is
$$\{x \in \mathbb{R}^n \ : \ Ax \le b\} \ .$$

- *Linear Programming:* Linear programming is the study of linear programs: modeling, formulation, algorithms, and analysis. A linear program is an optimization problem wherein one either minimizes or maximizes a linear objective function in a finite number of decision variables subject to a finite number of linear inequality and/or equality constraints.

- *Objective Function:* The objective function in an optimization problem is the real-valued function of the decision variables that is to be either minimized or maximized.

- *Explicit and Implicit Linear Constraints:* The explicit linear constraints are those that are explicitly stated in a given problem. The implicit constraints are those constraints that are part of the *natural* description of the phenomenon under study. These are typically bound constraints on the decision variables. For example, the width of an object is necessarily non-negative.

- *Standard Form:* Any LP having the representation
$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \le b, \ 0 \le x \end{array}$$
is said to be in standard form.

- *Optimal Value:* The optimal value of an optimization problem is the value of the objective function at an optimal solution.

- *Optimal Solution:* An optimal solution of the optimization problem
$$\min \{f(x) \ : \ x \in X \subset \mathbb{R}^n\}$$
is any point $x^* \in X$ such that
$$f(x^*) \le f(x) \quad \text{for all } x \in X.$$

- *Feasible Solution:* An feasible solution of the optimization problem

$$\min\{f(x) \,:\, x \in X \subset \mathbb{R}^n\}$$

  is any point $x \in X$.

- *Infeasible LP:* An infeasible LP is an LP whose constraint region, or, equivalently, feasible set, is empty.

- *Unbounded LP:* An unbounded LP is one for which there is a sequence of feasible points whose objective value diverges to $+\infty$ in the case of maximization, and diverges to $-\infty$ in the case of minimization.

Chapter 2

- *Slack Variables:* Slack variables are introduced into an LP formulation in order to turn linear inequalities into linear equalities. For example, the constraint region for an LP in standard form is $\{x \in \mathbb{R}^n \,:\, Ax \leq b, 0 \leq x\}$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system of inequalities is converted into system of equations involving $n + m$ variables by setting

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, 2, \ldots, m\ ,$$

  with $x_{n+i} \geq 0$ $i = 1, \ldots, m$. The new variables x_{n+i} are called slack variables.

- *Dictionary for an LP in Standard Form:* Given the LP in standard form,

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b,\ 0 \leq x \end{array} ,$$

  we define the initial dictionary for this LP to be the system

$$\begin{array}{lrll} \mathcal{D}_I & x_{n+i} = b_i & - & \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, 2, \ldots, m\ , \\ & z = & & \sum_{j=1}^{n} c_j x_j \end{array} .$$

  A dictionary for the this LP is any system of the form

$$\begin{array}{lrll} \mathcal{D}_B & x_i = \hat{b}_i & - & \sum_{j\in N}^{n} \hat{a}_{ij}x_j, \quad i \in B\ , \\ & z = \hat{z} & + & \sum_{j\in N}^{n} c_j x_j \end{array}$$

  having the same set of solutions as the system $\mathcal{D}_I$ and where the index sets B and N satisfy (a) $B \cup N = \{1, 2, \ldots, n + m\}$, (b) $B \cap N = \emptyset$, and (c) B contains precisely m elements.

- *Feasible Solution of an LP:* A feasible solution of an LP is any point in the feasible region for the LP.

- *Optimal Solution for an LP:* An optimal solution for an LP is any feasible solution whose objective value equals the optimal value of the LP.

- *Infeasible LP:* An infeasible LP is any LP whose feasible region is empty.

- *Unbounded LP:* An unbounded LP is any feasible LP whose optimal value is $+\infty$ if it is a maximization problem and $-\infty$ if it is a minimization problem.

Chapter 3

- *An LP with Feasible Origin:* An LP with feasible origin is any LP whose feasible region contains the origin. For an LP in standard form, this implies that $b_i \geq 0,\ i = 1, \ldots, m$.

- *Basic Rule for Choosing the Entering Variable:* Given a feasible dictionary, or, equivalently, a feasible tableau for an LP in standard form, the basic rule for choosing the variable to enter the basis is to choose any one of the currently non-basic variables whose cost row coefficient is positive.

- *Basic Rule for Choosing the Leaving Variable:* Given a feasible dictionary, or, equivalently, a feasible tableau for an LP in standard form, the basic rule for choosing the variable to leave the basis is to choose any one of the currently basic variables whose non-negativity places the greatest restriction on increasing the value of the entering variable.

- *Degeneracy:* Degeneracy occurs in the simplex algorithm occurs when a simplex pivot does not change either the current value of the objective or the point identified by the dictionary. A dictionary (or tableau) for an LP in standard for is said to be degenerate if one of the currently basic variables is assigned the value zero by the dictionary (or tableau).

- *Degenerate Basic Solution:* A basic feasible solution to an LP in standard for is any feasible solution identified by a feasible dictionary for the LP. Such a solution is said to be degenerate if one of the currently basic variables in the basic feasible solution has the value zero.

- *Degenerate Simplex Iteration:* A degenerate simplex iteration is any simplex iteration that does not change the value of the objective.

- *Cycling:* Cycling occurs in the simplex algorithm when a string of degenerate dictionaries is repeated over and over again infinitely often.

- *The Smallest Subscript Rule:* The smallest subscript rule is an anti-cycling rule for choosing the entering and leaving variables in the simplex algorithm. The rule states that the entering and leaving variables are chosen to have the smallest subscript from among all viable candidates for entering and leaving the basis.

- *Auxiliary Problem:* The auxiliary problem to an LP in standard for is an LP whose purpose is to determine the feasibility of the LP, and if feasible, to locate an initial basic feasible solution and its associated dictionary. If

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b,\ 0 \leq x \end{array}$$

  is the LP in standard form, then one possible auxiliary problem is the LP

$$\begin{array}{ll} \text{maximize} & x_0 \\ \text{subject to} & Ax - x_0 e \leq b,\ 0 \leq x, x_0 \end{array},$$

  where $e \in \mathbb{R}^m$ is the vector of all ones.

- *Two Phase Simplex Method:* The two phase simplex algorithm is the form of the simplex algorithm that is applied to LPs in standard form that do not have feasible origin.

Phase I: Apply the simplex algorithm to the auxiliary problem. Two outcomes are possible.
  (i) The optimal value is positive which implies that the LP is infeasible.
  (ii) The optimal value is zero and an initial basic feasible solution is determined.

Phase II: Apply the simplex algorithm initiated at the basic feasible solution provided in Phase I. Again, two outcomes are possible.
  (i) The LP is determined to be unbounded.
  (ii) An optimal basic feasible solution is determined.

- *The Fundamental Theorem of Linear Programming: Every LP has the following three properties:*
  (i) *If it has no optimal solution, then it is either infeasible or unbounded.*
  (ii) *If it has a feasible solution, then it has a basic feasible solution.*
  (iii) *If it is bounded, then it has an optimal basic feasible solution.*