Math 407A: Linear Optimization

Lecture 14

Math Dept, University of Washington

October 30, 2009
General Duality Theory

General Weak Duality theorem

Theorems of the Alternative
General Duality Theory

It is useful to have a more general duality theory than the one we have presented thus far.
General Duality Theory

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By *more general*, I mean a theory that allows one to compute a dual LP without first having to transform the problem into standard form.
General Duality Theory

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By *more general*, I mean a theory that allows one to compute a dual LP without first having to transform the problem into standard form.

The great advantage of doing this is that it allows the modeler to understand the nature of the dual variables in terms of the original problem statement and the original decision variables.
General Duality Theory

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The great advantage of doing this is that it allows the modeler to understand the nature of the dual variables in terms of the original problem statement and the original decision variables.

In our discussion we still need to make use of a *standard form* but it will be much more general and flexible than the standard form used so far.
Expanded Standard Form for General Duality Theory

\[ \mathcal{P} \quad \text{maximize} \quad \sum_{j=1}^{n} c_j x_j \]

subject to \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i \in I \]
\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i \in E \]
\[ 0 \leq x_j \quad j \in R \] .

Here the index sets \( I, E, \) and \( R \) are such that
\[ I \cap E = \emptyset, \quad I \cup E = \{1, 2, \ldots, m\}, \quad \text{and} \quad R \subset \{1, 2, \ldots, n\} . \]
# Primal-Dual Correspondences

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Primal-Dual Correspondences

\[ P \text{ maximize } \sum_{j=1}^{n} c_j x_j \]
\[ \text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i \in I \]
\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i \in E \]
\[ 0 \leq x_j \quad j \in R \]
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\[ F = \{1, 2, \ldots, n\} \setminus R = \text{the free variables.} \]
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\[ \mathcal{P} \text{ maximize } \sum_{j=1}^{n} c_j x_j \]
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\[0 \leq x_j \quad j \in R\]

\[F = \{1, 2, \ldots, n\} \setminus R = \text{the free variables.}\]

\[\mathcal{D} \text{ minimize}\]
Primal-Dual Correspondences

\[ \mathcal{P} \text{ maximize } \sum_{j=1}^{n} c_j x_j \]
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\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i \in E \]
\[ 0 \leq x_j \quad j \in R \]

\[ F = \{1, 2, \ldots, n\} \setminus R = \text{the free variables.} \]

\[ \mathcal{D} \text{ minimize } \sum_{i=1}^{m} b_i y_i \]
Primal-Dual Correspondences

\[ \mathcal{P} \quad \text{maximize} \quad \sum_{j=1}^{n} c_j x_j \]
subject to
\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i \in I \]
\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i \in E \]
\[ 0 \leq x_j \quad j \in R \]

\[ F = \{1, 2, \ldots, n\} \setminus R = \text{the free variables.} \]

\[ \mathcal{D} \quad \text{minimize} \quad \sum_{i=1}^{m} b_i y_i \]

\[ 0 \leq y_i \quad i \in I \]
Primal-Dual Correspondences

\[ \mathcal{P} \] maximize \[ \sum_{j=1}^{n} c_j x_j \]
subject to \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i \in \mathcal{I} \]
\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i \in \mathcal{E} \]
\[ 0 \leq x_j \quad j \in \mathcal{R} \]

\[ F = \{1, 2, \ldots, n\} \setminus \mathcal{R} = \text{the free variables.} \]

\[ \mathcal{D} \] minimize \[ \sum_{i=1}^{m} b_i y_i \]
subject to \[ 0 \leq y_i \quad i \in \mathcal{I} \]
Primal-Dual Correspondences

\[ P \text{ maximize } \sum_{j=1}^{n} c_j x_j \]

subject to

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i \in I \]
\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i \in E \]
\[ 0 \leq x_j \quad j \in R \]

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\[ D \text{ minimize } \sum_{i=1}^{m} b_i y_i \]

subject to

\[ \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad j \in R \]

\[ 0 \leq y_i \quad i \in I \]
Primal-Dual Correspondences

\[ \mathcal{P} \quad \text{maximize} \quad \sum_{j=1}^{n} c_j x_j \]
\[ \text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i \in I \]
\[ \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i \in E \]
\[ 0 \leq x_j, \quad j \in R \]

\[ F = \{1, 2, \ldots, n\} \setminus R = \text{the free variables.} \]

\[ \mathcal{D} \quad \text{minimize} \quad \sum_{i=1}^{m} b_i y_i \]
\[ \text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j \in R \]
\[ \sum_{i=1}^{m} a_{ij} y_i = c_j, \quad j \in F \]
\[ 0 \leq y_i, \quad i \in I \]
Example: General Duality

Compute the dual of the LP

\[
\begin{align*}
& \text{maximize} & & x_1 - 2x_2 + 3x_3 \\
& \text{subject to} & & 5x_1 + x_2 - 2x_3 \leq 8 \\
& & & -x_1 + 5x_2 + 8x_3 = 10 \\
& & & x_1 \leq 10, \ 0 \leq x_3
\end{align*}
\]
Example: General Duality

Compute the dual of the LP

maximize $x_1 - 2x_2 + 3x_3$
subject to $5x_1 + x_2 - 2x_3 \leq 8$
$-x_1 + 5x_2 + 8x_3 = 10$
$x_1 \leq 10, \ 0 \leq x_3$

$y_1 \geq 0$
Example: General Duality

Compute the dual of the LP

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\begin{align*}
\text{maximize} & \quad x_1 - 2x_2 + 3x_3 \\
\text{subject to} & \quad 5x_1 + x_2 - 2x_3 \leq 8 \\
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Example: General Duality

Compute the dual of the LP

maximize \( x_1 - 2x_2 + 3x_3 \)
subject to \( 5x_1 + x_2 - 2x_3 \leq 8 \)
\(-x_1 + 5x_2 + 8x_3 = 10\)
\( x_1 \leq 10 \)
\( 0 \leq x_3 \)
\( y_1 \geq 0 \)
\( y_2 \) free
Example: General Duality

Compute the dual of the LP

maximize $x_1 - 2x_2 + 3x_3$
subject to

$5x_1 + x_2 - 2x_3 \leq 8 \quad y_1 \geq 0$
$-x_1 + 5x_2 + 8x_3 = 10 \quad y_2 \text{ free}$
$x_1 \leq 10 \quad y_3 \geq 0$
$0 \leq x_3$
Example: General Duality

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\begin{align*}
\text{maximize} & \quad x_1 - 2x_2 + 3x_3 \\
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& \quad -x_1 + 5x_2 + 8x_3 = 10 \\
& \quad x_1 \leq 10 \\
& \quad 0 \leq x_3 \\
& \quad y_1 \geq 0 \\
& \quad y_2 \text{ free} \\
& \quad y_3 \geq 0
\end{align*}
\]

minimize
Example: General Duality

Compute the dual of the LP

maximize \( x_1 - 2x_2 + 3x_3 \)
subject to \( 5x_1 + x_2 - 2x_3 \leq 8 \)
\( -x_1 + 5x_2 + 8x_3 = 10 \)
\( x_1 \leq 10 \)
\( 0 \leq x_3 \)

\( y_1 \geq 0 \)
\( y_2 \) free
\( y_3 \geq 0 \)

minimize

\( 0 \leq y_1, 0 \leq y_3 \)
Example: General Duality

Compute the dual of the LP

\[
\begin{align*}
\text{maximize} & \quad x_1 - 2x_2 + 3x_3 \\
\text{subject to} & \quad 5x_1 + x_2 - 2x_3 \leq 8 \\
& \quad -x_1 + 5x_2 + 8x_3 = 10 \\
& \quad x_1 \leq 10 \\
& \quad 0 \leq x_3
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad 8y_1 + 10y_2 + 10y_3 \\
& \quad 0 \leq y_1, \ 0 \leq y_3
\end{align*}
\]
Example: General Duality

Compute the dual of the LP

\[
\begin{align*}
\text{maximize} & \quad x_1 - 2x_2 + 3x_3 \\
\text{subject to} & \quad 5x_1 + x_2 - 2x_3 \leq 8 \\
& \quad -x_1 + 5x_2 + 8x_3 = 10 \\
& \quad x_1 \leq 10 \\
& \quad 0 \leq x_3 \\
& \quad y_1 \geq 0 \\
& \quad y_2 \text{ free} \\
& \quad y_3 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad 8y_1 + 10y_2 + 10y_3 \\
\text{subject to} & \quad 0 \leq y_1, \ 0 \leq y_3
\end{align*}
\]
Example: General Duality

Compute the dual of the LP

maximize \( x_1 - 2x_2 + 3x_3 \)
subject to \( 5x_1 + x_2 - 2x_3 \leq 8 \quad y_1 \geq 0 \)
\( -x_1 + 5x_2 + 8x_3 = 10 \quad y_2 \) free
\( x_1 \leq 10 \quad y_3 \geq 0 \)
\( 0 \leq x_3 \)

minimize \( 8y_1 + 10y_2 + 10y_3 \)
subject to \( 5y_1 - y_2 + y_3 = 1 \)

\( 0 \leq y_1, 0 \leq y_3 \)
Example: General Duality

Compute the dual of the LP

\[
\begin{align*}
\text{maximize} & \quad x_1 - 2x_2 + 3x_3 \\
\text{subject to} & \quad 5x_1 + x_2 - 2x_3 \leq 8 \quad y_1 \geq 0 \\
& \quad -x_1 + 5x_2 + 8x_3 = 10 \quad y_2 \text{ free} \\
& \quad x_1 \leq 10 \quad y_3 \geq 0 \\
& \quad 0 \leq x_3
\end{align*}
\]

minimize \[ 8y_1 + 10y_2 + 10y_3 \]

subject to \[ 5y_1 - y_2 + y_3 = 1 \]
\[ y_1 + 5y_2 = -2 \]
\[ 0 \leq y_1, \ 0 \leq y_3 \]
Example: General Duality

Compute the dual of the LP

maximize \( x_1 - 2x_2 + 3x_3 \)
subject to
\[
\begin{align*}
5x_1 + x_2 - 2x_3 & \leq 8 & y_1 & \geq 0 \\
-x_1 + 5x_2 + 8x_3 & = 10 & y_2 & \text{free} \\
x_1 & \leq 10 & y_3 & \geq 0 \\
0 & \leq x_3
\end{align*}
\]

minimize \( 8y_1 + 10y_2 + 10y_3 \)
subject to
\[
\begin{align*}
5y_1 - y_2 + y_3 & = 1 \\
y_1 + 5y_2 & = -2 \\
-2y_1 + 8y_2 & \geq 3 \\
0 & \leq y_1, 0 & \leq y_3
\end{align*}
\]
Second Example: General Duality

maximize \quad 2x_1 - 3x_2 + x_3

subject to \quad x_1 + 5x_2 - 2x_3 = 4

10x_1 + x_2 - 5x_3 \leq 20

5x_1 - x_2 - x_3 = 3

x_1 \leq 6, \quad 0 \leq x_2
Second Example: Solution

Primal

maximize $2x_1 - 3x_2 + x_3$
subject to
$x_1 + 5x_2 - 2x_3 = 4$
$10x_1 + x_2 - 5x_3 \leq 20$
$5x_1 - x_2 - x_3 = 3$
$x_1 \leq 6, 0 \leq x_2$

Dual

minimize $4y_1 + 20y_2 + 3y_3 + 6y_4$
subject to
$y_1 + 10y_2 + 5y_3 + y_4 = 2$
$5y_1 + y_2 - y_3 \geq -3$
$-2y_1 - 5y_2 - y_3 = 1$
$0 \leq y_2, 0 \leq y_4$
General Weak Duality theorem

**Theorem:** Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T Ax \leq b^T y.$$ 

Moreover, the following statements hold.
General Weak Duality theorem

**Theorem:** Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ is feasible for $P$ and $y \in \mathbb{R}^m$ is feasible for $D$, then

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Moreover, the following statements hold.

(i) If $P$ is unbounded, then $D$ is infeasible.
General Weak Duality theorem

**Theorem:** Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ is feasible for $P$ and $y \in \mathbb{R}^m$ is feasible for $D$, then

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Moreover, the following statements hold.

(i) If $P$ is unbounded, then $D$ is infeasible.

(ii) If $D$ is unbounded, then $P$ is infeasible.
General Weak Duality theorem

**Theorem:** Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ is feasible for $P$ and $y \in \mathbb{R}^m$ is feasible for $D$, then

$$c^T x \leq y^T Ax \leq b^T y.$$

Moreover, the following statements hold.

(i) If $P$ is unbounded, then $D$ is infeasible.

(ii) If $D$ is unbounded, then $P$ is infeasible.

(iii) If $\bar{x}$ is feasible for $P$ and $\bar{y}$ is feasible for $D$ with $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is an optimal solution to $P$ and $\bar{y}$ is an optimal solution to $D$. 
General Weak Duality theorem

**Proof:** $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.
General Weak Duality theorem

**Proof:** \( x \in \mathbb{R}^n \) is feasible for \( P \) and \( y \in \mathbb{R}^m \) is feasible for \( D \).

\[
c^T x = \sum_{j \in R} c_j x_j + \sum_{j \in F} c_j x_j
\]
General Weak Duality theorem

Proof: $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

$$c^T x = \sum_{j \in R} c_j x_j + \sum_{j \in F} c_j x_j$$

$$\leq \sum_{j \in R} (\sum_{i=1}^{m} a_{ij} y_i) x_j$$

(Since $c_j \leq \sum_{i=1}^{n} a_{ij} y_i$ and $x_j \geq 0$ for $j \in R$.)
General Weak Duality theorem

Proof: $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

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$$\leq \sum_{j \in R} (\sum_{i=1}^m a_{ij} y_i) x_j + \sum_{j \in F} (\sum_{i=1}^m a_{ij} y_i) x_j$$

(Since $c_j \leq \sum_{i=1}^n a_{ij} y_i$ and $x_j \geq 0$ for $j \in R$
and $c_j = \sum_{i=1}^n a_{ij} y_i$ for $j \in F$.)
General Weak Duality theorem

**Proof:** $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

\[ c^T x = \sum_{j \in R} c_j x_j + \sum_{j \in F} c_j x_j \]

\[ \leq \sum_{j \in R} (\sum_{i=1}^{m} a_{ij} y_i) x_j + \sum_{j \in F} (\sum_{i=1}^{m} a_{ij} y_i) x_j \]

(Since $c_j \leq \sum_{i=1}^{n} a_{ij} y_i$ and $x_j \geq 0$ for $j \in R$

and $c_j = \sum_{i=1}^{n} a_{ij} y_i$ for $j \in F$.)

\[ = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} y_i x_j \]
General Weak Duality theorem

Proof: $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

$$c^T x = \sum_{j \in R} c_j x_j + \sum_{j \in F} c_j x_j$$

$$\leq \sum_{j \in R} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j + \sum_{j \in F} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j$$

(Since $c_j \leq \sum_{i=1}^{n} a_{ij} y_i$ and $x_j \geq 0$ for $j \in R$ and $c_j = \sum_{i=1}^{n} a_{ij} y_i$ for $j \in F$.)

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} y_i x_j$$

$$= y^T Ax$$
General Weak Duality theorem

\[ x^T Ay \]
General Weak Duality theorem

\[ x^T A y = \sum_{i \in I} (\sum_{j=1}^n a_{ij} x_j) y_i + \sum_{i \in E} (\sum_{j=1}^n a_{ij} x_j) y_i \]
General Weak Duality theorem

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\[ \leq \sum_{i \in I} b_i y_i \]

(Since \( \sum_{j=1}^{n} a_{ij} x_j \leq b_i \) and \( 0 \leq y_i \) for \( i \in I \))
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\[ = \sum_{i=1}^{m} b_i y_i \]

\[ = b^T y . \]
Systems of Equations and Inequalities

Let \( g \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \), and \( A \in \mathbb{R}^{m \times n} \).
Systems of Equations and Inequalities

Let $g \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$.

**Question:** Does there exist $x \in \mathbb{R}^n$ such that

$$0 \leq x, \quad g^T x < 0, \quad \text{and} \quad Ax = b?$$
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We answer this question by considering the following LP.
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\begin{align*}
\text{minimize} & \quad g^T x \\
\text{subject to} & \quad Ax = b, \quad 0 \leq x.
\end{align*}
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If the answer to the above question is Yes, then the optimal value in this LP is
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If the answer to the above question is Yes, then the optimal value in this LP is \(-\infty\).
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Let \( g \in \mathbb{R}^n, \ b \in \mathbb{R}^m, \) and \( A \in \mathbb{R}^{m \times n}. \)

**Question:** Does there exist \( x \in \mathbb{R}^n \) such that

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We answer this question by considering the following LP.

minimize \( g^T x \)

subject to \( Ax = b, \ 0 \leq x. \)

If the answer to the above question is Yes, then the optimal value in this LP is \(-\infty\).

What does this say about the dual to this LP?
Systems of Equations and Inequalities

The dual to the LP

\[
\text{maximize } \ -g^T x \\
\text{subject to } \ Ax = b, \ 0 \leq x
\]

is
Systems of Equations and Inequalities

The dual to the LP

\[
\begin{align*}
\text{maximize} & \quad -g^T x \\
\text{subject to} & \quad Ax = b, \ 0 \leq x
\end{align*}
\]

is

\[
\begin{align*}
\text{minimize} & \quad b^T x \\
\text{subject to} & \quad A^T y \geq -g, \ 0 \leq y
\end{align*}
\]
Systems of Equations and Inequalities

The dual to the LP

\[
\begin{align*}
&\text{maximize} \quad -g^T x \\
&\text{subject to} \quad Ax = b, \; 0 \leq x
\end{align*}
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is

\[
\begin{align*}
&\text{minimize} \quad b^T x \\
&\text{subject to} \quad A^T y \geq -g, \; 0 \leq y
\end{align*}
\]

What is the relationship between these two LPs?
A Theorem of the Alternative

**Theorem:** Either there exists a solution \( x \in \mathbb{R}^n \) to the system

\[
0 \leq x, \quad g^T x < 0, \quad \text{and} \quad Ax = b
\]

or there exists a solution \( y \in \mathbb{R}^m \) to the system

\[
0 \leq g + A^T y \quad \text{and} \quad 0 \leq y,
\]

but not both.