4 Duality Theory

We now dive deeply into the duality theory of linear programming. As we will see, the solution to the dual problem is most often just as important as the solution to the primal, and in some cases more important. Recall from Section 1 that the dual to an LP in standard form

$$(\mathcal{P}) \qquad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \le b,\ 0 \le x \end{array}$$

is the LP

$$(\mathcal{D}) \qquad \begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \ge c,\ 0 \le y. \end{array}$$

Since the problem $\mathcal{D}$ is a linear program, it too has a dual. The *duality* terminology suggests that the problems $\mathcal{P}$ and $\mathcal{D}$ come as a pair implying that the dual to $\mathcal{D}$ should be $\mathcal{P}$. This is indeed the case as we now show. Observe that by using standard techniques the dual problem can converted to standard form:

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \ge c, \\ & 0 \le y \end{array} \quad \overset{\substack{\text{standard} \\ \text{form}}}{\Longrightarrow} \quad \begin{array}{ll} -\text{maximize} & (-b)^T y \\ \text{subject to} & (-A^T) y \le (-c), \\ & 0 \le y. \end{array}$$

The problem on the right is in standard form so we can take its dual to get an LP which also can be written in standard form:

$$\begin{array}{ll} \text{minimize} & (-c)^T x \\ \text{subject to} & (-A^T)^T x \ge (-b),\ 0 \le x \end{array} \quad \overset{\substack{\text{standard} \\ \text{form}}}{\Longrightarrow} \quad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \le b,\ 0 \le x\ . \end{array}$$

Consequently, the dual of the dual is the primal.

Next recall that the primal-dual pair of LPs $\mathcal{P} - \mathcal{D}$ are related via the Weak Duality Theorem.

Theorem 4.1 (Weak Duality Theorem) *If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then*

$$c^T x \le y^T A x \le b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T\bar{x} = b^T\bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

We now use The Weak Duality Theorem in conjunction with The Fundamental Theorem of Linear Programming to prove the *Strong Duality Theorem of Linear Programming.* The key ingredient in this proof is the general form for simplex tableaus derived at the end of Section 2 in (2.5).

Theorem 4.2 (The Strong Duality Theorem of Linear Programming) *If either $\mathcal{P}$ or $\mathcal{D}$ has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions to both $\mathcal{P}$ and $\mathcal{D}$ exist.*

Remark: This result states that the finiteness of the optimal value implies the existence of a solution. This is not always the case for nonlinear optimization problems. Indeed, consider the problem

$$\min_{x \in \mathsf{R}} e^x.$$

This problem has a finite optimal value, namely zero; however, this value is not attained by any point $x \in \mathbb{R}$. That is, it has a finite optimal value, but a solution does not exist. The existence of solutions when the optimal value is finite is one of the many special properties of linear programs.

Proof: Since the dual of the dual is the primal, we may as well assume that the primal has a finite optimal value. In this case, the Fundamental Theorem of Linear Programming says that an optimal basic feasible solution exists. By our formula for the general form of simplex tableaus (2.5), we know that there exists a nonsingular record matrix $R \in \mathbb{R}^{n \times n}$ and a vector $y \in \mathbb{R}^m$ such that the optimal tableau has the form

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ c^T - y^T A & -y^T & -y^T b \end{bmatrix}.$$

Since this is an optimal tableau, we have

$$c - A^T y \leq 0, \qquad\qquad -y^T \leq 0$$

with $y^T b$ equal to optimal value in the primal problem. But then $A^T y \geq c$ and $0 \leq y$ so that y is feasible for the dual problem $\mathcal{D}$. In addition, the Weak Duality Theorem implies that

$$\begin{array}{rclll} b^T y & = & \text{maximize} & c^T x & \leq \; b^T \widehat{y} \\ & & \text{subject to} & Ax \leq b, \; 0 \leq x & \end{array}$$

for every vector $\widehat{y}$ that is feasible for $\mathcal{D}$. Therefore, y solves $\mathcal{D}$!!!! ■

This is an amazing fact! Our method for solving the primal problem $\mathcal{P}$, the simplex algorithm, simultaneously solves the dual problem $\mathcal{D}$! This fact is of enormous practical value when we study sensitivity analysis.

4.1 Complementary Slackness

The Strong Duality Theorem tells us that optimality is equivalent to equality in the Weak Duality Theorem. That is, x solves $\mathcal{P}$ and y solves $\mathcal{D}$ if and only if (x, y) is a $\mathcal{P} - \mathcal{D}$ feasible pair and

$$c^T x = y^T A x = b^T y.$$

We now carefully examine the consequences of this equivalence. Note that the equation $c^T x = y^T Ax$ implies that

$$0 = x^T(A^T y - c) = \sum_{j=1}^{n} x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j). \tag{4.1}$$

In addition, primal and dual feasibility implies that

$$0 \leq x_j \qquad \text{and} \qquad 0 \leq \sum_{i=1}^{m} a_{ij}y_i - c_j \qquad \text{for } j = 1, \ldots, n,$$

respectively, and so

$$x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j) \geq 0 \qquad \text{for} \qquad j = 1, \ldots, n.$$

Hence, the only way (4.1) can hold is if

$$x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j) = 0 \qquad \text{for} \qquad j = 1, \ldots, n.$$

or equivalently,

$$x_j = 0 \qquad \text{or} \qquad \sum_{i=1}^{m} a_{ij}y_i = c_j \qquad \text{or both for } j = 1, \ldots, n. \tag{4.2}$$

Similarly, $y^T Ax = y^T b$ tells us that

$$0 = y^T(b - Ax) = \sum_{i=1}^{m} y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j).$$

Again, and primal dual feasibility implies that

$$0 \leq y_i \qquad \text{and} \qquad 0 \leq b_i - \sum_{j=1}^{n} a_{ij}x_j \qquad \text{for } i = 1, \ldots, m,$$

respectively. Thus, we must have

$$y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j) = 0 \qquad \text{for} \qquad j = 1, \ldots, n,$$

or equivalently,

$$y_i = 0 \qquad \text{or} \qquad \sum_{j=1}^{n} a_{ij}x_j = b_i \qquad \text{or both for } i = 1, \ldots, m. \tag{4.3}$$

The two observations (4.2) and (4.3) combine to yield the following theorem.

Theorem 4.8 (The Complementary Slackness Theorem) *The vector $x \in \mathsf{R}^n$ solves $\mathcal{P}$ and the vector $y \in \mathsf{R}^m$ solves $\mathcal{D}$ if and only if x is feasible for $\mathcal{P}$ and y is feasible for $\mathcal{D}$ and*

(i) either $0 = x_j$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$ or both for $j = 1, \ldots, n$, and

(ii) either $0 = y_i$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$ or both for $i = 1, \ldots, m$.

Proof: If x solves $\mathcal{P}$ and y solves $\mathcal{D}$, then by the Strong Duality Theorem we have equality in the Weak Duality Theorem. But we have just observed that this implies (4.2) and (4.3) which are equivalent to (i) and (ii) above.

Conversely, if (i) and (ii) are satisfied, then we get equality in the Weak Duality Theorem. Therefore, by Theorem 4.2, x solves $\mathcal{P}$ and y solves $\mathcal{D}$. ■

The Complementary Slackness Theorem can be used to develop a test of optimality for a putative solution to $\mathcal{P}$ (or $\mathcal{D}$). We state this test as a corollary.

Corollary 4.1 *The vector $x \in \mathbb{R}^n$ solves $\mathcal{P}$ if and only if x is feasible for $\mathcal{P}$ and there exists a vector $y \in \mathsf{R}^m$ feasible for $\mathcal{D}$ such that*

(i) for each $i \in \{1, 2, \ldots, m\}$, if $\sum_{j=1}^{n} a_{ij} x_j < b_i$, then $y_i = 0$, and

(ii) for each $j \in \{1, 2, \ldots, n\}$, if $0 < x_j$, then $\sum_{i=1}^{m} a_{ij} y_i = c_j$.

Proof: (i) and (ii) implies equality in the Weak Duality Theorem. The primal feasibility of x and the dual feasibility of y combined with Theorem 4.1 yield the result. ■

We now show how to apply this Corollary to test whether or not a given point solves an LP. Recall that all of the nonbasic variables in an optimal BFS take the value zero, and, if the BFS is nondegenerate, then all of the basic variables are nonzero. That is, m of the variables in the optimal BFS are nonzero since every BFS has m basic variables. Consequently, among the n original decision variables and the m slack variables, m variables are nonzero at a nondegenerate optimal BFS. That is, among the constraints

$$\begin{array}{ll} 0 \le x_j & j = 1, \ldots, n, \\ 0 \le x_{n+i} = c_i - \sum_{i \in N} a_{ij} x_j & i = 1, \ldots, m \end{array}$$

m of them are strict inequalities. If we now look back at Corollary 4.1, we see that every nondegenerate optimal basic feasible solution yields a total of m equations that an optimal dual solution y must satisfy. That is, Corollary 4.1 tells us that the m optimal dual variables

y_i satisfy m equations. Therefore, we can write an $m \times m$ system of equations to solve for y. We illustrate this by applying Corollary 4.1 to the following LP

$$
\begin{array}{lrcrcrcrcrcl}
\text{maximize} & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & & \\
\text{subject to} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \le & 4 \\
 & 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \le & 3 \\
 & 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \le & 5 \\
 & 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \le & 1 \\
 & & & & & \multicolumn{3}{c}{0 \le x_1, x_2, x_3, x_4, x_5.} & & & &
\end{array} \tag{4.10}
$$

Does the point

$$x^T = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$$

solves this LP? Following Corollary 4.1, if x is optimal, then x must be feasible for (4.10) and there must exists a vector $y \in \mathbb{R}^4$ feasible for the dual LP to (4.10) and which satisfies the conditions given in items (i) and (ii) of the corollary. To check that x is primal feasible fist observe that x is componentwise positive. Next, by plugging x into the remaining constraints for (4.10) we see that equality is attained in each of the constraints except the third:

$$
\begin{array}{rcrcrcrcrcl}
(0) & + & 3\left(\frac{4}{3}\right) & + & 5\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 2(0) & = & 4 \\
4(0) & + & 2\left(\frac{4}{3}\right) & - & 2\left(\frac{2}{3}\right) & + & \left(\frac{5}{3}\right) & + & (0) & = & 3 \\
2(0) & + & 4\left(\frac{4}{3}\right) & + & 4\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 5(0) & < & 5 \\
3(0) & + & \left(\frac{4}{3}\right) & + & 2\left(\frac{2}{3}\right) & - & \left(\frac{5}{3}\right) & - & 2(0) & = & 1.
\end{array}
$$

Hence, x is primal feasible. Moreover, by item (i) of Corollary 4.1, we see that the vector $y \in \mathbb{R}^4$ that we seek must have

$$y_3 = 0 \tag{4.11}$$

due to the strict inequality in the associated primal constraint. Since $x_2 > 0$, $x_3 > 0$, and $x_4 > 0$, item (ii) of Corollary 4.1 implies that the vector y we are looking for must also satisfy the dual equalities

$$
\begin{array}{rcrcrcrcl}
3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & = & 6 \\
5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & = & 5 \\
-2y_1 & + & y_2 & - & 2y_3 & - & y_4 & = & -2.
\end{array} \tag{4.12}
$$

Putting (4.11) and (4.12) together, we see that y must satisfy

$$
\begin{bmatrix} 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -2 \\ 0 \end{pmatrix},
$$

where the first three rows come from (4.12) and the last row comes from (4.11). We reduce the associated augmented system as follows:

$$
\begin{array}{rrrr|r l}
3 & 2 & 4 & 1 & 6 & \\
5 & -2 & 4 & 2 & 5 & \\
-2 & 1 & -2 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
3 & 2 & 0 & 1 & 6 & r_1 - 4r_4 \\
5 & -2 & 0 & 2 & 5 & r_2 - 4r_4 \\
-2 & 1 & 0 & -1 & -2 & r_3 + 2r_4 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 3 & 0 & 0 & 4 & r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
0 & 3 & 0 & 0 & 3 & r_1 - r_2 \\
1 & 0 & 0 & 0 & 1 & \\
0 & 1 & 0 & -1 & 0 & r_3 + 2r_2 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & 1 & r_2 \\
0 & 1 & 0 & 0 & 1 & \frac{1}{3}r_1 \\
0 & 0 & 1 & 0 & 0 & r_4 \\
0 & 0 & 0 & 1 & 1 & -r_3 + \frac{1}{3}r_1
\end{array}
$$

This gives $y^T = (1, 1, 0, 1)$ as the only possible vector y that can satisfy the requirements of (i) and (ii) in Corollary 4.1. It remains to check that this y is dual feasible, that is, we need check that y is feasible for the dual LP to (4.10):

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
& 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
& 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
& -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
& 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
& & & \multicolumn{5}{l}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}
$$

Clearly, $0 \leq y$ and, by construction, the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality. Thus, it only remains to check the first and fifth inequalities:

$$
\begin{array}{rcrcrcrcl}
(1) & + & 4(1) & + & 2(0) & + & 3(1) & = & 8 \geq 7 \\
2(1) & + & (1) & + & 5(0) & - & 2(1) & = & 1 \ngeq 3.
\end{array}
$$

Therefore, y is not dual feasible. But as observed, this is the only possible vector y satisfying (i) and (ii) of Corollary (4.1), hence $x^T = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$ cannot be a solution to the LP (4.10).

4.2 General Duality Theory

Thus far we have discussed duality theory as it pertains to LPs in standard form. Of course, one can always transform any LP into one in standard form and then apply the duality theory. However, from the perspective of applications, this is cumbersome since it obscures the meaning of the dual variables. It is very useful to be able to compute the dual of an LP without first converting to standard form. In this section we show how this can easily be done. For this, we still make use of a standard form, but now we choose one that is much more flexible:

$$\begin{array}{lll} \mathcal{P} & \max & \sum_{j=1}^n c_j x_j & \\ & \text{subject to} & \sum_{j=1}^n a_{ij} x_j \le b_i & i \in I \\ & & \sum_{j=1}^n a_{ij} x_j = b_i & i \in E \\ & & 0 \le x_j & j \in R \quad . \end{array}$$

Here the index sets I, E, and R are such that

$$I \cap E = \emptyset, \ I \cup E = \{1, 2, \ldots, m\}, \text{ and } R \subset \{1, 2, \ldots, n\}.$$

We use the following primal-dual correspondences to compute the dual of an LP.

In the Dual	In the Primal
Restricted Variables	Inequality Constraints
Free Variables	Equality Constraints
Inequality Constraints	Restricted Variables
Equality Constraints	Free Variables

Using these rules we obtain the dual to $\mathcal{P}$:

$$\begin{array}{llll} \mathcal{D} & \min & \sum_{i=1}^m b_i y_i & \\ & \text{subject to} & \sum_{i=1}^m a_{ij} y_i \ge c_j & j \in R \\ & & \sum_{i=1}^m a_{ij} y_i = c_j & j \in F \\ & & 0 \le y_i & i \in I \quad , \end{array}$$

where $F = \{1, 2, \ldots, n\} \setminus R$.

For example, the LP

$$\begin{array}{lll} \text{maximize} & x_1 - 2x_2 + 3x_3 & \\ \text{subject to} & 5x_1 + x_2 - 2x_3 & \le 8 \\ & -x_1 + 5x_2 + 8x_3 & = 10 \\ & x_1 \le 10, \ 0 \le x_3 & \end{array}$$

has dual

$$\begin{array}{lll} \text{minimize} & 8y_1 + 10y_2 + 10y_3 & \\ \text{subject to} & 5y_1 - y_2 + y_3 & = 1 \\ & y_1 + 5y_2 & = -2 \\ & -2y_1 + 8y_2 & \ge 3 \\ & 0 \le y_1, \ 0 \le y_3 & . \end{array}$$

The primal-dual pair $\mathcal{P}$ and $\mathcal{D}$ above are related by the following weak duality theorem.

Theorem 4.9 *[General Weak Duality Theorem]*
Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T A x \leq b^T y.$$

Moreover, the following statements hold.

(i) If $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is infeasible.

(ii) If $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is infeasible.

(iii) If $\bar{x}$ is feasible for $\mathcal{P}$ and $\bar{y}$ is feasibe for $\mathcal{D}$ with $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is and optimal solution to $\mathcal{P}$ and $\bar{y}$ is an optimal solution to $\mathcal{D}$.

Proof: Suppose $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$. Then

$$\begin{aligned}
c^T x &= \sum_{j\in R} c_j x_j + \sum_{j\in F} c_j x_j \\
&\leq \sum_{j\in R} (\sum_{i=1}^m a_{ij} y_i) x_j + \sum_{j\in F} (\sum_{i=1}^m a_{ij} y_i) x_j \\
&\qquad\qquad \text{(Since } c_j \leq \textstyle\sum_{i=1}^m a_{ij} y_i \text{ and } x_j \geq 0 \text{ for } j \in R \\
&\qquad\qquad\quad \text{and } c_j = \textstyle\sum_{i=1}^m a_{ij} y_i \text{ for } j \in F.) \\
&= \sum_{i=1}^m \sum_{j=1}^n a_{ij} y_i x_j \\
&= y^T A x \\
&= \sum_{i\in I} (\sum_{j=1}^n a_{ij} x_j) y_i + \sum_{i\in E} (\sum_{j=1}^n a_{ij} x_j) y_i \\
&\leq \sum_{i\in I} b_i y_i + \sum_{i\in E} b_i y_i \\
&\qquad\qquad \text{(Since } \textstyle\sum_{j=1}^n a_{ij} x_j \leq b_i \text{ and } 0 \leq y_i \text{ for } i \in I \\
&\qquad\qquad\quad \text{and } \textstyle\sum_{j=1}^n a_{ij} x_j = b_i \text{ for } i \in E. \\
&= \sum_{i=1}^m b_i y_i \\
&= b^T y\ .
\end{aligned}$$

■

4.3 The Dual Simplex Algorithm

Consider the linear program

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 \\ & \text{subject to} & -x_1 - x_2 + 2x_3 \le -3 \\ & & -4x_1 - 2x_2 + x_3 \le -4 \\ & & x_1 + x_2 - 4x_3 \le 2 \\ & & 0 \le x_1, x_2, x_3 \ , \end{array}$$

and its dual

$$\begin{array}{lll} \mathcal{D} & \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ & \text{subject to} & -y_1 - 4y_2 + y_3 \ge -4 \\ & & -y_1 - 2y_2 + y_3 \ge -2 \\ & & 2y_1 + y_2 - 4y_3 \ge -1 \\ & & 0 \le y_1, y_2, y_3 \ . \end{array}$$

Problem $\mathcal{P}$ does not have feasible origin, and so it appears that one must apply Phase I of the two phase simplex algorithm to obtain an initial basic feasible solution. On the other hand, the dual problem $\mathcal{D}$ does have feasible origin. Is it possible to apply the simplex algorithm to $\mathcal{D}$ and avoid Phase I altogether? Yes, however, we do it in a way that may at first seem odd. We *reverse* the usual simplex procedure by choosing a pivot row first, and then choosing the pivot column. The initial tableau for the problem $\mathcal{P}$ is

x_1	x_2	x_3	x_4	x_5	x_6	
-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

A striking and important feature of this tableau is that every entry in the cost row is nonpositive! This is exactly what we are trying to achieve by our pivots in the simplex algorithm. This is a consequence of the fact that the dual problem $\mathcal{D}$ has feasible origin. Any tableau having this property we will call *dual feasible*. Unfortunately, the tableau is not feasible since some of the right hand sides are negative. Henceforth, we will say that such a tableau is not *primal feasible*. That is, instead of saying that a tableau (or dictionary) is feasible or infeasible in the usual sense, we will now say that the tableau is *primal feasible*, respectively, *primal infeasible*.

Observe that if a tableau is *both* primal and dual feasible, then it must be optimal, i.e. the basic feasible solution that it identifies is an optimal solution. We now describe an implementation of the simplex algorithm, called the *dual simplex algorithm*, that can be applied to tableaus that are dual feasible but not primal feasible. Essentially it is the simplex algorithm applied to the dual problem but using the tableau structure associated with the primal problem. The goal is to use simplex pivots to attain primal feasibility while maintaining dual feasibility.

Consider the tableau above. The right hand side coefficients are -3, -4, and 2. These correspond to the cost coefficients of the dual objective. Note that this tableau also identifies a basic feasible solution for the dual problem by setting the dual variable equal to the negative of the cost row coefficients associated with the slack variables:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

The dual variables are currently "nonbasic" and so their values are zero. Next note that by increasing the value of either y_1 or y_2 we decrease the value of the dual objective since the coefficients of these variables are -3 and -4. In the simplex algorithm terminology, we can pivot on either the first or second row to decrease the value of the dual objective. Let's choose the first row as our pivot row. How do we choose the pivot column? Similar to the primal simplex algorithm, we choose the pivot column to maintain dual feasibility. Let us have a look at what this means by examining the dual constraints:

$$\begin{array}{rrrl} -y_1 & -4y_2 & +y_3 & \geq -4 \\ -y_1 & -2y_2 & +y_3 & \geq -2 \\ 2y_1 & +y_2 & -4y_3 & \geq -1. \end{array} \tag{4.13}$$

If we plug $y = 0$ into these inequalities, we interpret the "dual slack" in the three equations as $r := A^T y - c = (4, 2, 1)^T$, while the dual variables are $y = (0, 0, 0)^T$. We call this the dual basic feasible solution associated with this dual feasible tableau, with the components of y being non-basic (having the value zero) and those of r basic. If we increase the value of y_1 from zero, we decrease the value of the dual objective since the y_1 coefficient in the dual objective is -3. The dual inequalities in (4.13) limit the amount by which we can increase y_1 and preserve dual feasibility, i.e. $r = A^T y - c \geq 0$ and $y \geq 0$. The first dual inequality in (4.13) limits the increase in y_1 to 4, the second limits the increase to 2, while the final inequality does not limit y_1 at all since the coefficient on y_1 is positive.

This process of computing the largest possible increase in a dual variable while maintaining dual feasibility corresponds the similar process in the primal simplex algorithm where we computed ratios and chose the minimum ratio. In the dual simplex algorithm we again must compute ratios, but this time it is the ratios of the negative entries in the pivot row with the corresponding cost row entries:

ratios for the first two columns are 4 and 2

-1	[-1]	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

The smallest ratio is 2 so the pivot column is column 2 in the tableau, and the pivot element is therefore the (1,2) entry of the tableau. Note that this process of choosing the pivot is the reverse of how the pivot is chosen in the primal simplex algorithm. In the dual simplex algorithm we first choose a pivot row, then compute ratios to determine the pivot column which identifies the pivot. We now successive apply this process to the above tableau until optimality is achieved.

-1	[-1]	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	[-2]	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	
1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	optimal

Therefore, the optimal solutions to $\mathcal{P}$ and $\mathcal{D}$ are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 0 \\ 5/2 \end{pmatrix},$$

respectively, with optimal value $z = -17/2$.

Next consider the LP

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 \\ & \text{subject to} & -x_1 - x_2 + 2x_3 \le -3 \\ & & -4x_1 - 2x_2 + x_3 \le -4 \\ & & x_1 + x_2 - x_3 \le 2 \\ & & 0 \le x_1, x_2, x_3 \ . \end{array}$$

This LP differs from the previous LP only in the x_3 coefficient of the third linear inequality. Let's apply the dual simplex algorithm to this LP.

-1	**[-1]**	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	

1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

The first dual simplex pivot is given above. Repeating this process again, we see that there is only one candidate for the pivot row in our dual simplex pivoting strategy. What do we do now? It seems as though we are stuck since there are no negative entries in the third row with which to compute ratios to determine the pivot column. What does this mean? Recall that we chose the pivot row because the negative entry in the right hand side implies that we can decrease the value of the dual objective by bring the dual variable y_3 into the dual basis. The ratios are computed to preserve dual feasibility. In this problem, the fact that there are no negative entries in the pivot row implies that we can increase the value of y_3 as much as we want without violating dual feasibility. That is, the dual problem is unbounded below, and so, by the weak duality theorem, the primal problem must be infeasible!

We will make extensive use of the dual simplex algorithm in our discussion of sensitivity analysis in linear programming.