Markowitz Mean-Variance Portfolio Theory
Exercise Solutions

1. (a) In this problem $\mathrm{m} = \mu\mathrm{e}$, and so the expected return for any set of weights w is $\mathrm{m}^T w = \mu \mathrm{e}^T w = 1$. Therefore, the efficient set must consist of a single point with expected return μ and variance equal to the minimum variance.
(b) The minimum variance solution is $w = \frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}}$. The associated variance is $w^T\Sigma w = (\mathrm{e}^T\Sigma^{-1}\mathrm{e})^{-1} = 1/\hat{\sigma}^2$. Therefore the minimum variance point on the efficient frontier, and which in fact equals the entire frontier, is $(\hat{\sigma}^{-1}, \mu)$.

2. There are two ways to interpret this problem:
   (A) The planner spends 1mil on the concert and spends more money on rain insurance. Here $0 \leq u \leq 3$.
   (B) The planner spends u million dollars on insurance and $(1-u)$ million dollars on the concert where $0 \leq u \leq 1$ since 1mil is the entire budget for the concert. For the concert to proceed, the planner needs partners to come up with the remaining u million dollars.

   Solutions for both scenarios are provided.

   (A) Let r be the rate of return random variable. If we buy u million dollars of insurance, then $r = \frac{Q-(1+u)}{1+u}$ where Q is the expected payout. Therefore,

$$r = \begin{cases} \frac{3-(u+1)}{(u+1)} & \text{, if no rain,} \\ \frac{2u-(u+1)}{(u+1)} & \text{, if rain.} \end{cases}$$

   The probability of each of the two events, *rain* and *no rain*, is 1/2, consequently

$$\begin{aligned} \mu_r &= E(r) = \frac{1}{2}\frac{3-(u+1)}{(u+1)} + \frac{1}{2}\frac{2u-(u+1)}{(u+1)} = \frac{1}{2(u+1)} \\ \sigma_r^2 &= E(r^2) - \mu_r^2 = \frac{4u^2-12u+9}{4(u+1)} \ . \end{aligned}$$

   We have

$$\frac{d\sigma_r^2}{du} = \frac{20u-30}{4(1+u)^3},$$

   so the minimum variance solution occurs at $u = 3/2$ at which point the variance of the rate of return is zero yielding a guarenteed rate of return of 1/5. That is, the planner should buy 1.5mil in insurance for a guarenteed payout of $3,000,000.

(B) In this scenario we have two investment opportunities: (1) the concert and (2) rain insurance. Let r_1 be the rate of return on the concert investmment and r_2 be the rate of return on the rain insurance:

$$r_1 = \left[\begin{array}{ll} \frac{3-1}{1} = 2 & \text{, if no rain,} \\ \frac{0-1}{1} = -1 & \text{, if rain.} \end{array} \right.$$

$$r_2 = \left[\begin{array}{ll} \frac{0-1}{1} = -1 & \text{, if no rain,} \\ \frac{2u-u}{u} = 1 & \text{, if rain.} \end{array} \right.$$

We spend the entire 1mil on a portfolio consisting of r_1 and r_2, and so this portfolio is

$$r = (1-u)r_1 + ur_2.$$

Consequently,

$$\begin{aligned} \mu_r &= (1-u)E(r_1) + uE(r_2) = \frac{1}{2}(1-u) \\ \sigma_r^2 &= (1-u)^2\sigma_1^2 + 2u(1-u)\sigma_{12} + u^2\sigma_2^2 = (1-u)^2\frac{9}{4} - 2u(1-u)\frac{3}{2} + u^2. \end{aligned}$$

To obtain the minimum variance portfolio solve

$$0 = \frac{d\sigma_r^2}{du} = \frac{1}{2}(25u - 15)$$

to get $u = \frac{3}{5}$. The corresponding expected return and variance are 1/5 and 0, respectively. That is, if we purchase \$600,000 and invest \$400,000 in the concert, then we get a guarenteed payout of \$1,200,000.

3. (a)

$$\Sigma^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

Hence the minimum variance weights are

$$\frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

yielding a minimum variance portfolio of $r = \frac{1}{2}r_1 + \frac{1}{2}r_2$.

(b) A second efficient portfolio is given by the weights

$$w_{mk} = \frac{\Sigma^{-1}\mathrm{m}}{\mathrm{m}^T\Sigma^{-1}\mathrm{e}} = \frac{1}{6}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

yielding the efficient portfolio $r = \frac{1}{6}(3r_1 - 2r_2 + 5r_3)$.

(c) The market portfolio in this setting has weights

$$w_{\mathrm{M}} = \begin{pmatrix} 1 - \mathrm{e}^T\Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \\ \Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \end{pmatrix} = \begin{pmatrix} .6 \\ .2 \\ -.2 \\ .4 \end{pmatrix} .$$

If desired, these weights can be combined with the weights $(1, 0, 0, 0)^T$ to eliminate the risk free component to get the efficient weights $\frac{1}{3}(0, 1, -1, 2)$ yielding the efficient portfolio $r = \frac{1}{3}(r_1 - r_2 + 2r_3)$.

4. (a) The random vector $\hat{x} = (r_{\mathrm{M}}, r_1, r_2, \ldots, r_n)^T$ has by definition the covariance matrix

$$\hat{\Sigma} = \begin{bmatrix} \sigma_{\mathrm{M}}^2 & s^T \\ s & \Sigma \end{bmatrix} .$$

Hence

$$\mathrm{var}(w_1 r_1 + \cdots + w_n r_n - r_{\mathrm{M}}) = \begin{pmatrix} -1 \\ w_1 \\ \vdots \\ w_n \end{pmatrix}^T \hat{\Sigma} \begin{pmatrix} -1 \\ w_1 \\ \vdots \\ w_n \end{pmatrix} = w^T\Sigma w - 2s^T w + \sigma_{\mathrm{M}}^2.$$

(b)

$$\begin{aligned} 0 &= \Sigma w - s - \gamma\mathrm{e} \\ 1 &= \mathrm{e}^T w \end{aligned}$$

(c)

$$w = \Sigma^{-1}s + \frac{1 - \mathrm{e}^T\Sigma^{-1}s}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}}\Sigma^{-1}\mathrm{e}$$

(d)(i)

$$\begin{aligned} 0 &= \Sigma w - s - \lambda\mathrm{m} - \gamma\mathrm{e} \\ 1 &= \mathrm{e}^T w \\ \mu_b &\le \mathrm{m}^T w \\ 0 &\le \lambda \\ 0 &= \lambda(\mathrm{m}^T w - \mu_b) \end{aligned}$$

(ii) The case where the constraint $\mu_b \le \mathrm{m}^T w$ is inactive has already been studied above, so we assume that $\mu_b = \mathrm{m}^T w$. In this case we get

$$w = \Sigma^{-1}s + (1 - \mathrm{e}^T\Sigma^{-1}s)\left[\frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}} + \alpha\left[\frac{\Sigma^{-1}\mathrm{m}}{\mathrm{e}^T\Sigma^{-1}\mathrm{m}} - \frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}}\right]\right] ,$$

where

$$\alpha = \frac{(e^T\Sigma^{-1}m)(e^T\Sigma^{-1}e)}{\delta}\left[(1-e^T\Sigma^{-1}s)^{-1}(\mu_b - m^T\Sigma^{-1}s) - (e^T\Sigma^{-1}m)\right]$$

and

$$\delta = (m^T\Sigma^{-1}m)(e^T\Sigma^{-1}e) - (e^T\Sigma^{-1}m)^2 \ .$$

(iii) Yes! The two funds are

$$\begin{aligned} w_1 &= \Sigma^{-1}s + \frac{1-e^T\Sigma^{-1}s}{e^T\Sigma^{-1}e}\Sigma^{-1}e \\ w_2 &= \Sigma^{-1}s + \frac{1-e^T\Sigma^{-1}s}{e^T\Sigma^{-1}m}\Sigma^{-1}m \end{aligned}$$

5. (a) $\mu = .07 + \frac{1}{2}\sigma$
(b) .64
(c) Borrow \$1000 at the risk free rate and invest \$2000 in the market portfolio.
(d) \$ 1182

6. (a) We have

$$\sigma_\alpha^2 = (1-\alpha)\sigma_m^2 + 2\alpha(1-\alpha)\sigma_{\tau m} + \alpha^2\sigma_\tau^2,$$

and since σ_m is the minimum variance portfolio we have

$$\begin{aligned} 0 &= \left.\frac{d\sigma_\alpha^2}{d\alpha}\right|_{\alpha=0} \\ &= 2(\sigma_{\tau m} - \sigma_m^2) \ . \end{aligned}$$

Therefore, $A = 1$, i.e.,$\sigma_{\tau m} = \sigma_m^2$.
(b) Just solve $0 = \text{cov}(w_z, w_\tau)$ for α to get $\alpha = \frac{\sigma_m^2}{\sigma_\tau^2 - \sigma_m^2}$.

7. Apply the CAPM formula to get

$$P = \frac{\mu_Q}{1 + r_f + \beta_r(\mu_m - r_f)} = \frac{1000}{1.1 + .6(.17 - .1)} = \frac{1000}{1.142} = 875.66 \ .$$

Hence according to the CAPM these shares are a bargin at \$800.