Solutions to Constrained Optimization Problems

(1) (a) The KKT conditions are

$$\begin{aligned}
(1)\qquad 20 &\geq e^{x_1} + e^{x_2},\ 0 \leq x_1 \\
(2)\qquad 0 &\leq \lambda,\ \mu \\
(3)\qquad 0 &= \lambda(e^{x_1} + e^{x_2} - 20),\ 0 = \mu x_1 \\
(4)\qquad 0 &= e^{(x_1 - x_2)} + \lambda e^{x_1} - \mu \\
(5)\qquad 0 &= -e^{(x_1 - x_2)} + \lambda e^{x_2}
\end{aligned}$$

Equation (5) implies that $\lambda = e^{(x_1 - 2x_2)} \neq 0$. Therefore, the first complementarity condition in (3) implies that $e^{x_1} + e^{x_2} = 20$. Adding equations (4) and (5) gives

$$0 = \lambda(e^{x_1} + e^{x_2}) - \mu = 20\lambda - \mu,$$

so that $\mu = 20\lambda \neq 0$. Therefore, the second complementarity condition (3) implies that $x_1 = 0$. Hence, $(x_1, x_2) = (0, \log 19)$ is the only KKT point. Note that this is a convex programming problem satisfying the Slater condition and so this KKT point is the unique global solution.

(b) The KKT conditions are

$$\begin{aligned}
(6)\qquad 20 &\geq e^{x_1} + e^{x_2},\ 0 \leq x_1 \\
(7)\qquad 0 &\leq \lambda,\ \mu \\
(8)\qquad 0 &= \lambda(e^{x_1} + e^{x_2} - 20),\ 0 = \mu x_1 \\
(9)\qquad 0 &= -e^{(x_2 - x_1)} + \lambda e^{x_1} - \mu \\
(10)\qquad 0 &= e^{(x_2 - x_1)} + \lambda e^{x_2}
\end{aligned}$$

By equation (10), $\lambda = -e^{-x_1} < 0$. But this contradicts the requirement that $0 \leq \lambda$. Therefore, no KKT point exists. Nonetheless this is also a convex problem satisfying the Slater condition. So what is wrong here?!

No solutions exists. The optimal value is 0 but is not achieved.

(d) The KKT conditions are $Ax = b$ and $x = A^T y$ for some $y \in \mathbb{R}^m$. Multiplying the second condition through by A gives $b = Ax = AA^T y$. Since $\mathrm{Nul}(A^T) = \{0\}$ the matrix AA^T is invertable (Why?). Hence $\lambda = (AA^T)^{-1} b$ and $x = A^T (AA^T)^{-1} b$. Since this is a convex problem with a polyhedral constraint region, this unique KKT point is the unique global solution.

(2) Done in class.

(3) $\Omega = \left\{ (x_1, x_2)^T \mid x_1^2 \leq x_2, \ 0 \leq x_2 \right\}$

(4) (a) Since Ω is of the form given in equation (1) of the notes, we know that

$$T_\Omega(x) \subset \{ d \mid A_{i\cdot} d \leq 0, \ \text{for } i \in I(x), \ Ed = 0 \} .$$

Now given $d \in \{ d \mid A_{i\cdot} d \leq 0, \ \text{for } i \in I(x), \ Ed = 0 \}$, consider points of the form $x + td$ for $0 \leq t$. For such points we have

$$\begin{aligned} A_{i\cdot}(x + td) &= A_{i\cdot}x + tA_{i\cdot}d \\ &\leq A_{i\cdot}x \\ &\leq b \end{aligned}$$

for all $i \in I(x)$ and $t \geq 0$, and

$$\begin{aligned} E(x + td) &= Ex + tEd \\ &= Ex \\ &= h \end{aligned}$$

for all $t \geq 0$. For $i \in \{1, \ldots, m\}$ but $i \notin I(x)$, we know that $A_{i\cdot}x < b$. Hence there is a $\bar{t} > 0$ such that $A_{i\cdot}(x + td) < b$ for all $0 \leq t \leq \bar{t}$ for all $i \in \{1, \ldots, m\}$ but $i \notin I(x)$. Therefore, d is a feasible direction for Ω and so must be in $T_\Omega(x)$.

(b) We just showed this in Part (a) above.

(c) The set

$$\bigcup_{\lambda \geq 0} \lambda(\Omega - x) = \{ \lambda(y - x) \mid 0 \leq \lambda, \ y \in \Omega \}$$

is the set of feasible directions for a convex set since for every $y \in \Omega$ and $0 \leq \lambda \leq 1$ we have $x + \lambda(y - x) = (1 - \lambda)x + \lambda y \in \Omega$.

(5) Since $\nabla_x(\frac{1}{2}\|x - z\|) = x - z$, we have from Theorem 4.2 that $\bar{x}$ solves $\mathcal{D}$ if and only if

$$0 \leq (\nabla_x \tfrac{1}{2}\|\bar{x} - z\|)^T(x - \bar{x}) = (\bar{x} - z)^T(x - \bar{x}) \quad \forall \ x \in \Omega.$$