FINAL EXAM GUIDE

Calculators are not allowed for this exam. The exam will consist of 6 questions. Each question is worth 60 points except for question 4 which is worth 50 points. This gives a total of 350 points. The content of each question is as follows.

Question 1: (60 points) In this question you will be asked to model a given problem as an either an integer LP or a binary (0-1) LP. In addition, you may be asked how to model certain integer or 0-1 constraints. The models may include problems on graphs.

Question 2: (60 points) This question concerns the branch and bound algorithm. In particular, how does one interpret and prune a branch and bound tree, and how is the branch and bound algorithm implement to solve integer LPs.

Question 3: (60 points) This question concerns LP and integer LP reprocessing, as well as the construction of additional logical constraints in 0-1 LP.

Question 4: (50 points) This question will be about unimodularity, total unimodularity, and their implications for integer programming. In this context, knowledge of the properties and use of determinants and Cramer's Rule will be helpful, as well as the result on necessary conditions for total unimodularity and its proof.

Question 5: (60 points) This problem concerns the elementary properties of graphs and digraphs, specifically concerning walks, paths, cycles, cuts, trees, and vertex-edge (node-arc) incidence matrices. For example, showing that every walk between two vertices must contain a path where no two vertices are repeated, or establishing one of the tree equivalences. In addition, you need to know the relationship between spanning trees and connectedness as well as the algebraic properties of trees.

Question 6: (60 points) In this problem you will be given a capacitated graph and asked to solve the associated max-flow problem using the flow augmentation algorithm, and report both the optimal flow and the associated max-cut.

SAMPLE QUESTIONS

(1) The LED Company must draw up a production schedule for the next 9 weeks. Jobs last several weeks and once started must continue without interruption until completion. During each week a certain number of skilled workers are required to work full time on the job. Thus, if job i lasts p_i weeks, $l_{i,j}$ workers are required in week j for $j = 1, 2, \dots, p_i$ while the job is in progress. The total number of workers available in week t is L_t, $t = 1, 2, \dots, 9$. Typical job data are as follows:

Job	p_i	l_{i1}	l_{i2}	l_{i3}	l_{i4}
1	3	2	3	1	-
2	2	4	5	-	-
3	4	2	4	1	5
4	4	3	4	2	2
5	3	9	2	3	-

Assume that the are n jobs to be completed. Formulate the problem of finding a feasible schedule as an ILP.

(2) Branch and Bound

(a) Apply two more steps of the branch and bound algorithm for solving the following integer LP with associated optimal tableau.

$$\begin{array}{lrcrcrcl} \text{maximize} & 3x & + & 2y & - & 4z & & \\ \text{subject to} & x & + & 4y & & & \le & 5 \\ & 2x & + & 4y & - & 2z & \le & 5 \\ & x & + & y & - & 2z & \le & 2 \\ & 0 & \le & x, & y, & z & & \end{array}$$

$$\begin{array}{rrrrrr|r} 0 & 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1/2 & -1 & 1/2 \\ 1 & 3 & 0 & 0 & 1 & -1 & 3 \\ \hline 0 & -3 & 0 & 0 & -1 & -1 & -7 \end{array}$$

(b) Solve the following integer LP both graphically and by the branch and bound algorithm.

$$\begin{array}{lll} \text{maximize} & 9x_1 + 5x_2 & \\ \text{subject to} & 4x_1 + 9x_2 & \leq 35 \\ & 3x_1 + 2x_2 & \leq 19 \\ & x_1 - 3x_2 & \geq 1 \\ & x_1 & \leq 6 \\ & x \in \mathbb{Z}^2_+ & \end{array}$$

(3) (a) Use preprocessing to simplify the following LP.

$$\begin{array}{lrl} \text{maximize} & 2x_1 + x_2 - x_3 & \\ \text{subject to} & 5x_1 - 2x_2 + 8x_3 & \leq 15 \\ & 8x_1 + 3x_2 - x_3 & \geq 9 \\ & x_1 + x_2 + x_3 & \leq 6 \\ & 0 \leq x_1 \leq 1 & \\ & 0 \leq x_2 \leq 1 & \\ & 1 \leq x_3 & \\ & x \in \mathbb{Z}^3 & \end{array}$$

(b) Determine additional *logical inequalities* to simplify the following 0-1 programming problem.

$$\begin{array}{lrl} \text{maximize} & x_1 + 2x_2 - 2x_3 + x_4 & \\ \text{subject to} & 7x_1 + 3x_2 - 4x_3 - 2x_4 & \leq 1 \\ & -2x_1 + 7x_2 + 3x_3 + x_4 & \leq 6 \\ & -2x_2 - 3x_3 - 6x_4 & \leq -5 \\ & 3x_1 + x_2 - 3x_3 + x_4 & \geq -1 \\ & x \in \{0,1\}^4 \ . & \end{array}$$

(4) Total unimodularity.

(a) If $M \in \mathbb{Z}^{n \times n}$ is unimodular and $b \in \mathbb{Z}^m$, show that every solution to the system $Mx = b$ is integral.

(b) Assuming $A \in \mathbb{Z}^{m \times n}$ is totally unimodular, show that $[A\ I]$ is totally unimodular where I is the $n \times n$ identity matrix.

(c) If $A \in \mathbb{Z}^{m \times n}$ is totally unimodular, $c \in \mathbb{Z}^n$, and $b \in \mathbb{Z}^m$, why is it true that if the LP $\min\{c^T x : Ax \leq b,\ 0 \leq x\}$ has a solution, then it must have an integral solution.

(d) Determine if the following matrix is totally unimodular and justify your answer.

$$\begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(5) (a) Let $G = (V, E)$ be a graph with n vertices. Show that G is a tree if and only if G is connected and every edge in G is a bridge.

(b) Show that a graph is connected if and only if it contains a spanning tree.

(6) Apply the flow augmenting path algorithm to solve the max-flow problem given by the capacitated digraphs below. In this graph the source node is the node number 1 and the sink node is the highest numbered node. The arc capacities are the numbers associated with each arc. Initialize the flow at the zero flow. Report both the optimal flow and the associated optimal min cut.