MATH 516 SPRING 2007
FOURTH HOMEWORK SET

Home Due Tuesday 5/29/07

1. In this problem we study the theory and algorithmic implications for an interior point approach to the *horizontal linear complementarity problem*. In what follows we assume that the matrix $Q \in \mathbb{R}^{p\times p}$ is symmetric and positive semi-definite and that $c \in \mathbb{R}^p$, $A \in \mathbb{R}^{m\times p}$, $E \in \mathbb{R}^{k\times p}$, $d \in \mathbb{R}^k$, and $b \in \mathbb{R}^m$.

   (a) Assume that $\mathrm{Nul}\,(E^T) = \{0\}$ and consider the QP

   $$\begin{array}{lll} \mathcal{Q}_2 & \text{minimize} & \frac{1}{2}u^TQu - c^tu \\ & \text{subject to} & Au \leq b,\ Eu = d,\ 0 \leq u\ . \end{array}$$

   Related to $\mathcal{Q}_2$ is the so-called *horizontal LCP*

   The Horizontal LCP (HLCP)

   Given $M \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{k\times n}$, $h \in \mathbb{R}^k$, and $q \in \mathbb{R}^n$, find $x \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, and $y \in \mathbb{R}^n$ such that

   $$Mx + G^Tz + q = y,\ Gx = h,\ 0 \leq x,\ 0 \leq y,\ \text{and}\ x^Ty = 0\ .$$

   i. Write the KKT conditions for $\mathcal{Q}_2$.
   ii. Show that the KKT conditions for $\mathcal{Q}_2$ are an instance of the HLCP by specifying $M,\ G,\ h$, and q in terms of $Q,\ A,\ E,\ c,\ b$, and d.
   iii. Show that under this specification $\mathrm{Nul}\,(E^T) = \{0\}$ if and only if $\mathrm{Nul}\,(G^T) = \{0\}$.
   iv. Again, show that under this specification the positive semi-definiteness of Q implies that M is positive semi-definite.

   (b) Define

   $$F(x, z, y) = \begin{bmatrix} Mx + G^Tz - y + q \\ Gx - h \\ XYe \end{bmatrix}.$$

   i. Show that (x, z, y) solves the HLCP if and only if $F(x, z, y) = 0$ and $0 \leq x, y$.
   ii. Assume that M is positive semi-definite and $\mathrm{Nul}\,(G^T) = \{0\}$. Show that $F'(x, z, y)$ is nonsingular whenever $0 < x$ and $0 < y$.
   iii. Define

   $$\mathcal{F}_+ = \{(x, z, y) : Mx + G^Tz + q = y,\ Gx = h,\ 0 < x,\ 0 < y\}.$$

   Under the assumption that M is positive semi-definite, $\mathrm{Nul}\,(G^T) = \{0\}$, and $\mathcal{F}_+ \neq \emptyset$, show that the set

   $$\mathcal{F}(t) = \{(x, z, y) : Mx + G^Tz + q = y,\ Gx = h,\ 0 \leq x,\ 0 \leq y,\ x^Ty \leq t\}$$

   is compact for all $t \geq 0$.
   (*Hint*: There are a number of ways to show this. The best way to start is to first show that for all $(x, z, y) \in \mathcal{F}(t)$ (x, y) is bounded in 1-norm. This is done using exactly the same kind of argument as is used in the LCP case. To show that the z component is also bounded there are again a number of possible arguments. The most direct argument uses the fact that the matrix GG^T is invertible (see the midterm exam).)

   (c) Define

   $$\hat{\mathcal{F}}_+ = \{(x, y) : \exists z \in \mathbb{R}^k \text{ such that } (x, z, y) \in \mathcal{F}_+\}.$$

   Assume that M is positive semi-definite and $\mathrm{Nul}\,(G^T) = \{0\}$, and $\hat{\mathcal{F}}_+ \neq \emptyset$. Consider the mapping $u : \hat{\mathcal{F}}_+ \to \mathrm{int}\,(\mathbb{R}^n_+)$ given by

   $$u(x, y) = XYe$$

   .

i. Show that to each $(x, y) \in \hat{\mathcal{F}}_+$ there exists a unique $z \in \mathbb{R}^k$ such that $(x, z, y) \in \mathcal{F}_+$.

ii. Show that for every $a \in \operatorname{int}(\mathbb{R}^n_+)$ there exists a unique $(x, y) \in \hat{\mathcal{F}}_+$ such that $u(x, y) = a$. Using this fact one can show that u defines a diffeomorphism between $\hat{\mathcal{F}}_+$ and $\operatorname{int}(\mathbb{R}^n_+)$.

iii. Define a notion of *central path* for the HLCP and use the previous result to show that it exists under the given hypotheses.

iv. Show that a solution the the HLCP exists.

2. Program the *practical* infeasible interior point algorithm described in the notes and test it on the following test problems.

   Test Problems:

   > The matrix M is computed as follows: Let $A, B \in \mathbb{R}^{n\times n}$ and $q, d \in \mathbb{R}^n$ be randomly generated such that $a_{ij}, b_{ij} \in (-5, 5), q_i \in (-500, 500), d_i \in (0.0, 0.3)$ and that B is skew-symmetric. Define $M = A^T A + B + \operatorname{diag}(d)$. Generate ten problems in this way for each of the dimensions $n =$ 50, 100, 150, 200. Report the maximum, average, and minimum number of iterations needed by the algorithm to solve these problems. Also report back graphs of per–iteration data on a random problem that allow you to discuss the performance of the algorithm. Can you improve the performance of the algorithm by changing some of the parameters appearing in the algorithm?