MATH 516
SEVENTH HOMEWORK SET

(1) Program the *practical* infeasible interior point algorithm described in the notes and test it on the following test problems.

Test Problems:

> The matrix M is computed as follows: Let $A, B \in \mathbb{R}^{n\times n}$ and $q, d \in \mathbb{R}^n$ be randomly generated such that $a_{ij}, b_{ij} \in (-5,5), q_i \in (-500,500), d_i \in (0.0, 0.3)$. Define $M = A^TA + \frac{1}{2}(B - B^T) + \text{diag}\,(d)$. Generate ten problems in this way for each of the dimensions $n = 50,\ 100,\ 150,\ 200$. Report the maximum, average, and minimum number of iterations needed by the algorithm to solve these problems. Also report back graphs of per–iteration data on a random problem that allow you to discuss the performance of the algorithm. Can you improve the performance of the algorithm by changing some of the parameters appearing in the algorithm?

(2) In this problem we study the theory and algorithmic implications for an interior point approach to the *horizontal linear complementarity problem.* In what follows we assume that the matrix $Q \in \mathbb{R}^{p\times p}$ is symmetric and positive semi-definite and that $c \in \mathbb{R}^p$, $A \in \mathbb{R}^{m\times p}$, $E \in \mathbb{R}^{k\times p}$, $d \in \mathbb{R}^k$, and $b \in \mathbb{R}^m$.

(a) Assume that $\text{Nul}\,(E^T) = \{0\}$ and consider the QP

$$\begin{array}{lll} \mathcal{Q}_2 & \text{minimize} & \frac{1}{2}u^TQu - c^tu \\ & \text{subject to} & Au \le b,\ Eu = d,\ 0 \le u\ . \end{array}$$

Related to $\mathcal{Q}_2$ is the so-called *horizontal LCP*

The Horizontal LCP (HLCP)

Given $M \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{k\times n}$, $h \in \mathbb{R}^k$, and $q \in \mathbb{R}^n$, find $x \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, and $y \in \mathbb{R}^n$ such that

$$Mx + G^Tz + q = y,\ Gx = h,\ 0 \le x,\ 0 \le y,\ \text{and}\ x^Ty = 0\ .$$

(i) Write the KKT conditions for $\mathcal{Q}_2$.
(ii) Show that the KKT conditions for $\mathcal{Q}_2$ are an instance of the HLCP by specifying $M,\ G,\ h$, and q in terms of $Q,\ A,\ E,\ c,\ b$, and d.
(iii) Show that under this specification $\text{Nul}\,(E^T) = \{0\}$ if and only if $\text{Nul}\,(G^T) = \{0\}$.
(iv) Again, show that under this specification the positive semi-definiteness of Q implies that M is positive semi-definite.

(b) Define

$$F(x, z, y) = \begin{bmatrix} Mx + G^Tz - y + q \\ Gx - h \\ XYe \end{bmatrix}\ .$$

(i) Show that (x, z, y) solves the HLCP if and only if $F(x, z, y) = 0$ and $0 \le x, y$.
(ii) Assume that M is positive semi-definite and $\text{Nul}\,(G^T) = \{0\}$. Show that $F'(x, z, y)$ is nonsingular whenever $0 < x$ and $0 < y$.
(iii) Define

$$\mathcal{F}_+ = \{(x, z, y)\ :\ Mx + G^Tz + q = y,\ Gx = h,\ 0 < x,\ 0 < y\}.$$

Under the assumption that M is positive semi-definite, $\text{Nul}\,(G^T) = \{0\}$, and $\mathcal{F}_+ \ne \emptyset$, show that the set

$$\mathcal{F}(t) = \{(x, z, y)\ :\ Mx + G^Tz + q = y,\ Gx = h,\ 0 \le x,\ 0 \le y,\ x^Ty \le t\}$$

is compact for all $t \ge 0$.

(*Hint*: There are a number of ways to show this. The best way to start is to first show that for all $(x, z, y) \in \mathcal{F}(t)$ (x, y) is bounded in 1-norm. This is done using exactly the same kind of argument as is used in the LCP case. To show that the z component is also bounded there are again a number of possible arguments. The most direct argument uses the fact that the matrix GG^T is invertible (Why?).)

(c) Define

$$\hat{\mathcal{F}}_+ = \{(x, y) \,:\, \exists z \in \mathbb{R}^k \text{ such that } (x, z, y) \in \mathcal{F}_+\}.$$

Assume that M is positive semi-definite and $\mathrm{Nul}\,(G^T) = \{0\}$, and $\hat{\mathcal{F}}_+ \neq \emptyset$. Consider the mapping $u : \hat{\mathcal{F}}_+ \to \mathrm{int}\,(\mathbb{R}^n_+)$ given by

$$u(x, y) = XYe$$

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(i) Show that to each $(x, y) \in \hat{\mathcal{F}}_+$ there exists a unique $z \in \mathbb{R}^k$ such that $(x, z, y) \in \mathcal{F}_+$.

(ii) Show that for every $a \in \mathrm{int}\,(\mathbb{R}^n_+)$ there exists a unique $(x, y) \in \hat{\mathcal{F}}_+$ such that $u(x, y) = a$. Using this fact one can show that u defines a diffeomorphism between $\hat{\mathcal{F}}_+$ and $\mathrm{int}\,(\mathbb{R}^n_+)$.

(iii) Define a notion of *central path* for the HLCP and use the previous result to show that it exists under the given hypotheses.

(iv) Show that a solution the the HLCP exists.