AMATH/MATH 516
FIFTH HOMEWORK SET

(1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and $S \subset \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$. Given $x^0 \in \mathbb{R}^n$, show that if $\bar{x} \in \mathbb{R}^n$ is a local solution to $\min \left\{ f(x) \;\middle|\; x \in x^0 + S \right\}$, then

$$(*) \qquad \nabla f(x) \perp S \ .$$

(2) Let $Q \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$, with Q symmetric and positive definite, and consider the optimization problem $\min \left\{ \frac{1}{2}x^TQx + c^Tx \mid Ax \le b \right\}$ and its relaxation

$$\mathcal{R} \qquad \min \left\{ \frac{1}{2}x^TQx + c^Tx - t\sum_{i=1}^{n} \ln(y_i) \mid Ax + y = b \right\} \ ,$$

where $t > 0$ and we define $\ln(\mu) = -\infty$ if $\mu \le 0$.

(a) Use the optimality condition $(*)$ to show that the optimality conditions for $\mathcal{R}$ can be written as

$$(**) \qquad \exists\, y, w \in \mathbb{R}^m_+ \ \text{ s.t. } \ Ax + y = b,\ Qx + A^Tw + c = 0 \text{ and } \mathrm{diag}(w)\mathrm{diag}(y)\mathbf{1} = t\mathbf{1} \ ,$$

where $\mathbf{1}$ is always the vector of all ones of the appropriate dimension.

(b) Show that if (x^k, y^k, w^k, t_k) is a sequence of points satisfying $(**)$ with $t_k \downarrow 0$, then every cluster point of this sequence $(\bar{x}, \bar{y}, \bar{w}, 0)$ is such that $\bar{x}$ solves $\min \left\{ \frac{1}{2}x^TQx + c^Tx \mid Ax \le b \right\}$.

(3) Let $Q \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, and let $c \in \mathbb{R}^n$. Consider the optimization problem

$$\min_{0\le x} \frac{1}{2}x^TQx + c^Tx \ .$$

(a) What is the Lagrangian function for this problem?

(b) Show that the Lagrangian dual is the problem

$$\max_{y\le c} -\frac{1}{2}y^TQ^{-1}y \qquad = \qquad -\min_{y\le c} \frac{1}{2}y^TQ^{-1}y \ .$$

(c) Show that if $\bar{x}, \bar{y} \in \mathbb{R}^n$ satisfy $\bar{y} = -Q\bar{x}$, then $\bar{x}$ solves the primal problem if and only if $\bar{y}$ solves the dual problem, and the optimal values in the primal and dual coincide.

(4) Let $Q \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Consider the optimization problem

$$\mathcal{P} \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^TQx + c^Tx \\ \text{subject to} & \|x\|_\infty \le 1 \ . \end{array}$$

(a) Show that this problem is equivalent to the problem

$$\hat{\mathcal{P}} \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^TQx + c^Tx \\ \text{subject to} & -e \le x \le e \ , \end{array}$$

where e is the vector of all ones.

(b) What is the Lagrangian for $\hat{\mathcal{P}}$?

(c) Show that the Lagrangian dual for $\hat{\mathcal{P}}$ is the problem

$$\mathcal{D} \qquad \max\ -\frac{1}{2}(y-c)^TQ^{-1}(y-c) - \|y\|_1 \qquad = \qquad -\min \frac{1}{2}(y-c)^TQ^{-1}(y-c) + \|y\|_1 \ .$$

This is also the Lagrangian dual for $\mathcal{P}$.

(d) Show that if $\bar{x}, \bar{y} \in \mathbb{R}^n$ satisfy $\bar{y} = Q\bar{x} + c$, then $\bar{x}$ solves $\mathcal{P}$ if and only if $\bar{y}$ solves $\mathcal{D}$, and the optimal values in $\mathcal{P}$ and $\mathcal{D}$ coincide.

(5) Let $K \subset \mathbb{R}^m$ be a non-empty closed convex cone.

(a) If $K = \mathbb{R}^s_- \times \{0\}^{m-s}$, show that for every $x \in K$, $N(x \,|\, K) = \{ y \in \mathbb{R}^m \mid 0 \le y_i,\ y_ix_i = 0,\ i = 1, \ldots, s \}$.

(b) Show that, in general, $N(x \,|\, K) = \{ y \in K^\circ \mid \langle x, y\rangle = 0 \}$.

(c) Show that $\mathrm{dist}\,(x \,|\, K) = [\delta^*(\cdot \,|\, \mathbb{B}^\circ) \,\square\, \delta(\cdot \,|\, K)](x)$, that is, $\mathrm{dist}\,(x \,|\, K)$ is the infimal convolution of $\delta^*(\cdot \,|\, \mathbb{B}^\circ)$ and $\delta(\cdot \,|\, K)$, where $\mathbb{B}$ is the unit ball of the norm defining $\mathrm{dist}\,(x \,|\, K) := \inf \{ \|x - y\| \mid y \in K \}$.

(d) Given $f_1, f_2 \in \Gamma(\mathbb{R}^n)$, set $f = f_1 \,\square\, f_2$. Show that $f^* = f_1^* + f_2^*$, where

$$[f_1 \,\square\, f_2](x) := \inf \{ f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x \} \ .$$

(e) Use the previous two parts of this problem to show that $\mathrm{dist}\,(x \,|\, K) = \delta^*(x \,|\, \mathrm{B}^\circ \cap K^\circ)$ by using the fact that $f = f^{**}$ for all $f \in \Gamma(\mathbb{R}^n)$.

(f) Given $x \in K$, show that $\partial \mathrm{dist}\,(x \,|\, K) = \mathrm{B}^\circ \cap N(x \,|\, K)$.