

## CONVERGENCE PROPERTIES OF TRUST REGION METHODS FOR LINEAR AND CONVEX CONSTRAINTS

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We develop a convergence theory for convex and linearly constrained trust region methods which only requires that the step between iterates produce a sufficient reduction in the trust region subproblem. Global convergence is established for general convex constraints while the local analysis is for linearly constrained problems. The main local result establishes that if the sequence converges to a nondegenerate stationary point then the active constraints at the solution are identified in a finite number of iterations. As a consequence of the identification properties, we develop rate of convergence results by assuming that the step is a truncated Newton method. Our development is mainly geometrical; this approach allows the development of a convergence theory without any linear independence assumptions.

*Key words:* Trust region, linear constraints, convex constraints, global convergence, local convergence, degeneracy, rate of convergence, identification of active constraints, Newton's method, sequential quadratic programming, gradient projection.

### 1. Introduction

Each iteration of a trust region method for a linearly constrained problem requires the solution of a subproblem of the form

$$\min\{\psi_k(w): x_k + w \in \Omega, \|w\| \leq \Delta_k\}, \quad (1.1)$$

where  $\psi_k$  is a quadratic model of the reduction in the function,  $\Omega$  is the feasible set, the constraint  $x_k + w \in \Omega$  guarantees that the iterates remain feasible, and the constraint  $\|w\| \leq \Delta_k$  is the trust region. Convergence results for unconstrained trust region methods only require that the approximate solution  $s_k$  of subproblem (1.1) produce a sufficient reduction in the model  $\psi_k$ , and thus it is natural to search for an extension of the sufficient reduction concept to the constrained case. A sufficient reduction is certainly obtained if  $s_k$  is the global solution of subproblem (1.1), but this choice of  $s_k$  is not computationally realistic for a general quadratic  $\psi_k$ . The

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global solution of (1.1) can be obtained if we force  $\psi_k$  to be strictly convex, but this is contrary to the basic philosophy of trust region methods which requires  $\psi_k$  to be a model of the reduction in the function.

We develop a convergence theory for trust region methods which avoids convexity assumptions on the model and the need to obtain the global solution of the subproblem. Our development is mainly geometrical; this approach allows the development of a convergence theory without any linear independence assumptions. We show that global convergence can be established for the general minimization problem

$$\min\{f(x): x \in \Omega\}, \quad (1.2)$$

where  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuously differentiable mapping on the closed convex set  $\Omega$ . Local convergence results are mainly concerned with the polyhedral case where  $\Omega$  is defined by general linear constraints.

The aim of a global convergence analysis is to show that if  $\{x_k\}$  is the sequence generated by the trust region method, then every limit point of the sequence is a stationary point for problem (1.2). In this paper we follow Moré (1988) by developing the global convergence properties of the trust region method in terms of the projected gradient. This approach leads to stronger results.

The development of the *identification properties* of trust region methods is the crucial ingredient in the local convergence analysis. This requires proving that under the appropriate assumptions, if the sequence  $\{x_k\}$  converges to some  $x^*$  then there is an integer  $k_0$  such that the active constraints at  $x_k$  agree with the active constraints at  $x^*$  for all  $k \geq k_0$ .

Our development of global and local convergence results follows the outline that we have described. The development starts in Section 2 with some background material on projected gradients. This section presents the background material needed to establish the identification properties of trust region methods. Most of the results in this section can be found in the papers of Calamai and Moré (1987) and Burke and Moré (1988).

The trust region method for problem (1.2) is presented in Section 3. The development of this algorithm can be traced back to the work of Fletcher (1972) and Gay (1984) on general linearly constrained problems, the work of Conn, Gould and Toint (1988a, 1988b) on problem (1.2) when  $\Omega$  is defined by bound constraints, and the work of Toint (1988) on problems with a general closed convex  $\Omega$ . This algorithm is also related to the projected Newton method of Bertsekas (1982) for bound constrained problems, and to the two-metric projection algorithm of Gafni and Bertsekas (1984) for general linearly constrained problems. These projection algorithms, however, need convexity assumptions and anti-zigzagging strategies that are not needed by the trust region method.

The trust region method presented in Section 3 is based on the algorithm proposed by Toint (1988) with the modifications of the step suggested by Moré (1988). The main requirement is that the step produces a sufficient decrease in the model. There

is no need to obtain the global solution of an indefinite quadratic programming problem. Moreover, the requirements on the step can be satisfied by solving positive definite quadratic programming problems.

Section 4 considers the basic global convergence theory for the trust region method on a general closed convex set  $\Omega$ . The main result of this section is Theorem 4.4. This result extends the global convergence result of Powell (1984) for the unconstrained version of problem (1.2) where  $\Omega$  is  $\mathbb{R}^n$ . Theorem 4.4 also improves on the work of Toint (1988) for closed convex  $\Omega$  and on the work of Fletcher (1972, 1987) for linearly constrained problems because Toint assumes that  $\nabla f$  is Lipschitz continuous and that the feasible set  $\Omega$  is bounded, while Fletcher requires that the step be a global solution of subproblem (1.1).

The assumptions made by Theorem 4.4 on the model  $\psi_k$  are satisfied, for example, if  $\psi_k$  is the quadratic

$$\psi_k(w) = \langle \nabla f(w_k), w \rangle + \frac{1}{2} \langle w, B_k w \rangle, \tag{1.3}$$

and the matrices  $B_k$  satisfy the growth condition

$$\|B_k\| \leq \gamma k, \quad k \geq k_0, \tag{1.4}$$

for some constant  $\gamma$  and index  $k_0$ . If the model  $\psi_k$  satisfies these assumptions, and if the gradient  $\nabla f$  is uniformly continuous on the level set

$$\mathcal{L}(x_0) = \{x \in \Omega : f(x) \leq f(x_0)\},$$

then Theorem 4.4 implies that the trust region method of Section 3 generates a sequence  $\{x_k\}$  such that either  $\{f(x_k)\}$  is unbounded below or that

$$\liminf_{k \rightarrow \infty} \|x_k - P(x_k - \nabla f(x_k))\| = 0, \tag{1.5}$$

where  $P: \mathbb{R}^n \rightarrow \Omega$  is the projection into  $\Omega$ . If the sequence  $\{x_k\}$  is bounded then (1.5) implies that  $\{x_k\}$  has a limit point which is a stationary point of problem (1.2).

The case where  $\{f(x_k)\}$  is unbounded below can be avoided by assuming that  $\{x_k\}$  has a limit point or that  $f$  is bounded below on the level set  $\mathcal{L}(x_0)$ . These two assumptions are certainly satisfied if  $\Omega$  is bounded.

Section 5 considers convergence results in terms of the projected gradient  $\nabla_{\Omega} f$ . These results are motivated by the work of Calamai and Moré (1987) and Burke and Moré (1988), which show that the behavior of the sequence of projected gradients is closely related to the identification properties of the algorithm. Under the same assumptions as in Theorem 4.4 we show that for each iterate  $x_k$  it is possible to define a *Cauchy point*  $x_k^C$  such that

$$\liminf_{k \rightarrow \infty} \|\nabla_{\Omega} f(x_k^C)\| = 0. \tag{1.6}$$

If the sequence  $\{x_k\}$  is bounded then (1.6) also implies that  $\{x_k\}$  has a limit point which is a stationary point of problem (1.2). Also note that (1.6) applies, for example, to the *hypercube* method of Fletcher (1972) where  $\psi_k$  is the quadratic (1.3) and the

matrices  $B_k$  are generated by a quasi-Newton update which satisfies the growth condition (1.4).

In Section 5 we also develop stronger results which imply that every limit point of the sequence  $\{x_k\}$  is stationary. These results require additional assumptions on the model which are satisfied, for example, if  $\psi_k$  is the quadratic (1.3) and the sequence  $\{B_k\}$  is bounded. Under these assumptions on the model, Theorem 5.4 shows that if  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable on  $\Omega$  and  $x^*$  is a limit point of  $\{x_k\}$ , then there is a subsequence  $\{x_{k_i}\}$  which converges to  $x^*$  with

$$\lim_{i \rightarrow \infty} \|\nabla_{\Omega} f(x_{k_i}^C)\| = 0.$$

Moreover,  $\{x_{k_i}^C\}$  also converges to  $x^*$ , and thus  $x^*$  is a stationary point for problem (1.2).

Sections 6 and 7 contain the local convergence analysis of the trust region method; Section 6 is concerned with conditions which guarantee convergence of the iterates, while Section 7 develops the identification properties of the trust region method.

Most of the results in Sections 6 and 7 are concerned with the behavior of a trust region method in a neighborhood of a nondegenerate stationary point  $x^*$ , that is,

$$-\nabla f(x^*) \in \text{ri}(N(x^*)),$$

where  $N(x^*)$  is the normal cone at  $x^*$  and  $\text{ri}(\cdot)$  denotes the relative interior of a convex set. This definition of nondegeneracy is due to Dunn (1987). An advantage of this definition is that it does not make any linear independence assumptions on the constraints. We also note that if  $\Omega$  is polyhedral then  $x^*$  is nondegenerate if there is a set of positive Lagrange multipliers. This definition of nondegeneracy can thus be viewed as a generalization of the standard strict complementary condition.

In addition to nondegeneracy, results in Sections 6 and 7 usually assume that  $\Omega$  is polyhedral so that problem (1.2) is then a general linearly constrained problem, and that the approximate solution  $s_k$  of subproblem (1.1) is such that

$$\mathcal{A}(x_k^C) \subset \mathcal{A}(x_k + s_k), \tag{1.7}$$

where  $\mathcal{A}(x)$  is the set of active constraints at any  $x \in \Omega$ . This assumption on  $s_k$  can be satisfied, for example, by setting  $s_k = x_k^C - x_k$ .

Corollary 6.7 is typical of the convergence results of Sections 6. This result is concerned with variations on Newton's method, that is, methods whose model  $\psi_k$  is the quadratic (1.3) with  $B_k = \nabla^2 f(x_k)$ . We assume that  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is twice continuously differentiable on a polyhedral  $\Omega$ , that  $\nabla^2 f$  is bounded on the level set  $\mathcal{L}(x_0)$ , and that  $s_k$  satisfies (1.7). Corollary 6.7 shows that if  $\{x_k\}$  has a limit point  $x^*$  which is nondegenerate and satisfies

$$w \in N(x^*)^\perp, w \neq 0 \implies \langle w, \nabla^2 f(x^*) w \rangle > 0, \tag{1.8}$$

then  $\{x_k\}$  converges to  $x^*$ . Note that if  $x^*$  is nondegenerate then condition (1.8) is equivalent to the standard second order sufficiency condition. Results similar to Corollary 6.7 have been obtained by Conn, Gould and Toint (1988a) under the

assumption that  $\Omega$  is defined by bound constraints, and by Fletcher (1987) under the assumption that  $s_k$  is a global solution of subproblem (1.1) and that the active constraint normals are linearly independent.

The identification results of Section 7 can also be illustrated by considering variations on Newton’s method. We again assume that  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is twice continuously differentiable on a polyhedral  $\Omega$ , that  $\nabla^2 f$  is bounded on the level set  $\mathcal{L}(x_0)$ , and that  $s_k$  satisfies (1.7). Theorem 7.2 shows that if  $\{x_k\}$  converges to a nondegenerate point  $x^*$  then there is an index  $k_0 > 0$  such that

$$\mathcal{A}(x_k) = \mathcal{A}(x^*), \quad s_k \in N(x^*)^\perp, \quad k \geq k_0.$$

This result shows that the error  $x_k - x^*$  and the step  $s_k$  eventually belong to the subspace  $N(x^*)^\perp$ . In particular, all the iterates  $x_k$  and the trial iterates  $x_k + s_k$  eventually belong to an affine subspace, and thus once this subspace is identified, the algorithm is essentially unconstrained. Note that in Theorem 7.2 there is no need to assume condition (1.8); instead we assume that the sequence  $\{x_k\}$  converges to  $x^*$ .

Section 7 also develops rate of convergence results as a consequence of the identification properties. Rate of convergence results require further assumptions on  $s_k$ . Note that a superlinear rate of convergence is not a consequence of Theorem 7.2 because  $s_k$  is only required to produce a sufficient reduction in the sense of Section 3 and to satisfy (1.7). We develop rate of convergence results by assuming that  $s_k$  is a truncated Newton method.

We conclude by noting that our results on the identification properties of trust region methods compare favorably with previous results. Fletcher (1987) assumes linear independence of the active constraints and that  $s_k$  is a global minimizer of the subproblem (1.1). The status of the identification results obtained by Conn, Gould and Toint (1988a) for  $\Omega$  defined by bound constraints was uncertain because there was a gap in the proof of their Theorem 14. Conn, Gould and Toint have now filled this gap and a paper with the correction has been submitted for publication.

## 2. Projected gradients

A worthwhile feature of an optimization algorithm is the ability to identify the optimal active constraints at a stationary point. In this section we introduce those results that are needed to establish this identification property for trust region methods. Our approach is based in the notion of a projected gradient.

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable on a closed convex set  $\Omega$ , and let  $\nabla f$  be the gradient of  $f$  with respect to an inner product  $\langle \cdot, \cdot \rangle$ . Recall that a direction  $v$  is *feasible* at  $x \in \Omega$  if  $x + \tau v$  belongs to  $\Omega$  for all  $\tau > 0$  sufficiently small, and that the *tangent cone*  $T(x)$  is the closure of the cone of all feasible directions. The *projected gradient*  $\nabla_{\Omega} f$  of  $f$  is defined by

$$\nabla_{\Omega} f(x) \equiv \arg \min \{ \|v + \nabla f(x)\| : v \in T(x) \},$$

where the norm  $\|\cdot\|$  is generated by the inner product  $\langle \cdot, \cdot \rangle$ . Since  $T(x)$  is a nonempty closed convex set,  $\nabla_{\Omega}f(x)$  is uniquely defined.

A point  $x^* \in \Omega$  is a *stationary point* of problem (1.2) if and only if  $\nabla_{\Omega}f(x^*) = 0$ . An equivalent characterization of a stationary point is to require that

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad x \in \Omega.$$

This is the standard first order condition for a minimizer of  $f$ .

In a linearly constrained problem  $\Omega$  is a *polyhedral* set. There is no loss of generality in assuming that in a linearly constrained problem  $\Omega$  is defined by the set of linear constraints

$$\Omega = \{x \in \mathbb{R}^n : \langle c_j, x \rangle \geq \delta_j, j = 1, \dots, m\}, \tag{2.1}$$

for some vectors  $c_j \in \mathbb{R}^n$  and scalar  $\delta_j$ . In the linearly constrained case  $x^* \in \Omega$  is a stationary point if and only if  $x^*$  is a Kuhn-Tucker point. Thus,

$$\nabla f(x^*) = \sum_{j \in \mathcal{A}(x^*)} \lambda_j^* c_j, \quad \lambda_j^* \geq 0, \tag{2.2}$$

where the set of *active constraints* is defined by

$$\mathcal{A}(x) \equiv \{j : \langle c_j, x \rangle = \delta_j\}.$$

The tangent cone and the projected gradient can be expressed in terms of the active set. A computation shows that

$$T(x) = \{v \in \mathbb{R}^n : \langle c_j, v \rangle \geq 0, j \in \mathcal{A}(x)\}.$$

Moreover, Calamai and Moré (1987) prove that the Moreau decomposition of  $-\nabla f$  (see, for example, Zarantonello, 1971, Lemma 2.2) leads to the representation

$$\nabla_{\Omega}f(x) = -\nabla f(x) + \sum_{j \in \mathcal{A}(x)} \lambda_j c_j,$$

where  $\lambda_j$  for  $j \in \mathcal{A}(x)$  solves the bound constrained linear least squares problem

$$\min \left\{ \left\| \nabla f(x) - \sum_{j \in \mathcal{A}(x)} \lambda_j c_j \right\| : \lambda_j \geq 0 \right\}.$$

Note, in particular, that this representation of the projected gradient yields a unique value even if the active constraints  $c_j$  with  $j \in \mathcal{A}(x)$  are linearly dependent.

Let us now return to the case where  $\Omega$  is a general convex set. The following result of Calamai and Moré (1987) gives some of the basic properties of the projected gradient.

**Lemma 2.1.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable on the convex set  $\Omega$ .*

(a) *The point  $x \in \Omega$  is a stationary point of problem (1.2) if and only if  $\nabla_{\Omega}f(x) = 0$ .*

(b)  $\min\{\langle \nabla f(x), v \rangle : v \in T(x), \|v\| \leq 1\} = -\|\nabla_{\Omega}f(x)\|$ .

(c) *If  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable on  $\Omega$  then the mapping  $\|\nabla_{\Omega}f(\cdot)\|$  is lower semicontinuous on  $\Omega$ .  $\square$*

The term projected gradient is not entirely appropriate because at an interior point  $x$  of  $\Omega$  the projected gradient reduces to  $-\nabla f(x)$ . Part (b) of Lemma 2.1 shows that it might be more appropriate to call  $\nabla_{\Omega} f(x)$  the *projected steepest descent direction*. As noted by Calamai and Moré (1987), parts (a) and (c) of Lemma 2.1 imply, in particular, that if  $\{x_k\}$  is a sequence in  $\Omega$  which converges to  $x^*$ , and if  $\{\nabla_{\Omega} f(x_k)\}$  converges to zero, then  $x^*$  is a stationary point of problem (1.2).

We now consider the identification property at nondegenerate stationary points. The definition of nondegeneracy is expressed in terms of the dual or *normal cone*  $N(x)$  of the tangent cone  $T(x)$  where the *dual* of a cone  $K$  is the set of vectors  $v$  such that  $\langle v, w \rangle \leq 0$  for all  $w \in K$ . Stationary points can be defined in terms of the normal cone because  $x^*$  is a stationary point of problem (1.2) if and only if  $-\nabla f(x^*)$  belongs to  $N(x^*)$ . Also note that if  $\Omega$  is the polyhedral set defined by (2.1) then the Farkas lemma implies that  $N(x)$  is the cone generated by the active constraints normal  $-c_i$ , that is,

$$N(x) = \left\{ v \in \mathbb{R}^n : v = - \sum_{j \in \mathcal{A}(x)} \lambda_j c_j, \lambda_j \geq 0 \right\}. \tag{2.3}$$

In the following definition  $\text{ri}(\cdot)$  denotes the *relative interior* of a convex set, that is, the interior relative to the affine hull of the set.

**Definition.** The stationary point  $x^*$  is *nondegenerate* if  $-\nabla f(x^*) \in \text{ri}(N(x^*))$ .

If  $\Omega$  is the polyhedral convex set defined by (2.1) then Burke and Moré (1988) show that  $x^*$  is a nondegenerate stationary point if and only if there is a set  $\lambda_j^* > 0$  which satisfies (2.2). For this reason, the nondegeneracy assumption is sometimes called a *strict complementarity condition*. Note that we have not made any linear independence assumptions on the active constraints. If the active constraints are linearly dependent then there is an infinite number of multiplier sets  $\{\lambda_j^*\}$  which satisfy (2.2); nondegeneracy only requires the existence of one set of positive multipliers. The following result relates the projected gradient to the identification property.

**Theorem 2.2.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on a polyhedral  $\Omega$ , and let  $\{x_k\}$  be an arbitrary sequence in  $\Omega$  which converges to  $x^*$ . If  $\{\nabla_{\Omega} f(x_k)\}$  converges to zero, and  $x^*$  is nondegenerate, then  $\mathcal{A}(x_k) = \mathcal{A}(x^*)$  for all  $k \geq 0$  sufficiently large.  $\square$*

Calamai and Moré (1987) established Theorem 2.2 under the assumption that the active constraint normals were linearly independent. Burke and Moré (1988) were able to drop this assumption and to extend this result to certain non-polyhedral sets  $\Omega$ .

Identification properties can be studied in terms of the faces of the polyhedron  $\Omega$ . If the polyhedron  $\Omega$  is defined by (2.1) and  $\mathcal{A}$  is a set of constraints, then a *face*  $\Omega_F$  is defined by

$$\Omega_F = \{x \in \Omega : \langle c_j, x \rangle = \delta_j, j \in \mathcal{A}\}. \tag{2.4}$$

A short computation shows that the relative interior of the face  $\Omega_F$  is given by

$$\text{ri}(\Omega_F) = \{x \in \mathbb{R}^n : \langle c_j, x \rangle = \delta_j, j \in \mathcal{A}, \langle c_j, x \rangle > \delta_j, j \notin \mathcal{A}\}. \quad (2.5)$$

The relative interior of the faces of  $\Omega$  form a partition of  $\Omega$  because any  $x \in \Omega$  belongs to the relative interior of precisely one face. The relative interior of a face can also be identified with an active set by noting that  $x \in \text{ri}(\Omega_F)$  if and only if (2.5) holds with  $\mathcal{A} = \mathcal{A}(x)$ .

The explicit representation (2.5) of the relative interior of a face shows that the set of active constraints  $\mathcal{A}(x)$  is independent of  $x \in \text{ri}(\Omega_F)$ . As a consequence, the normal cone  $N(x)$  is independent of  $x \in \text{ri}(\Omega_F)$ . We define the *normal cone*  $N(\Omega_F)$  as the normal cone for any  $x \in \text{ri}(\Omega_F)$ . The following special case of a result of Burke and Moré (1988) provides an important property of faces in polyhedral sets.

**Theorem 2.3.** *If  $\Omega_F$  is a face of a polyhedral  $\Omega$  then  $\Omega_F + N(\Omega_F)$  has an interior and thus*

$$\text{int}\{\Omega_F + N(\Omega_F)\} = \text{ri}\{\Omega_F + N(\Omega_F)\} = \text{ri}(\Omega_F) + \text{ri}(N(\Omega_F)). \quad \square$$

Burke and Moré (1988) showed that Theorem 2.3 plays a key role in the identification properties of gradient projection and sequential quadratic programming algorithms. In particular, they derived Theorem 2.2 as a consequence of this result. We shall show that Theorem 2.3 is also applicable to trust region methods.

Our development of the convergence properties for trust region methods requires knowledge of a few basic properties of projection operators. One of the properties of projection operators that we need is that  $P(x)$  can be characterized in terms of the inner product by requiring that

$$\langle P(x) - x, P(x) - z \rangle \leq 0, \quad z \in \Omega. \quad (2.6)$$

In terms of normal cones, this characterization just requires that  $x - P(x)$  belongs to the normal cone  $N[P(x)]$ . Another relationship between projections and normal cones is obtained by noting that (2.6) implies that

$$N(x) = \{v \in \mathbb{R}^n : P(x + v) = x\} \quad (2.7)$$

for  $x \in \Omega$ . Our analysis also requires the following monotonicity properties.

**Lemma 2.4.** *If  $P$  is the projection into  $\Omega$  then the function  $\phi_1$  defined by*

$$\phi_1(\alpha) = \|P(x + \alpha d) - x\|, \quad \alpha > 0,$$

*is isotone (nondecreasing) for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{R}^n$ , and the function  $\phi_2$  defined by*

$$\phi_2(\alpha) = \|P(x + \alpha d) - x\|/\alpha, \quad \alpha > 0,$$

*is antitone (nonincreasing) for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{R}^n$ .  $\square$*



A proof of the isonicity of  $\phi_1$  can be found in Toint (1988). The antitonicity of  $\phi_2$  is due to Gafni and Bertsekas (1984); an alternate proof of this result can be found in Calamai and Moré (1987). A useful corollary of Lemma 2.4 is that for any  $\gamma > 0$ ,

$$\|P(x + \alpha_2 d) - x\| \geq \min\{\gamma, 1\} \|P(x + \alpha_1 d) - x\|, \quad \alpha_2 \geq \gamma \alpha_1 > 0. \tag{2.8}$$

This follows from Lemma 2.4 because if  $\hat{\gamma} = \min\{\gamma, 1\}$  then

$$\phi_1(\alpha_2) \geq \phi_1(\hat{\gamma} \alpha_1) = \hat{\gamma} \alpha_1 \phi_2(\hat{\gamma} \alpha_1) \geq \hat{\gamma} \alpha_1 \phi_2(\alpha_1) = \hat{\gamma} \phi_1(\alpha_1).$$

We conclude this section by noting a relationship between the projected gradient and the directional derivative of the mapping  $\alpha \mapsto P(x - \alpha \nabla f(x))$  at  $\alpha = 0$ .

**Lemma 2.5.** *If  $P$  is the projection into  $\Omega$  and  $x \in \Omega$  then*

$$\lim_{\alpha \rightarrow 0^+} \frac{P(x - \alpha \nabla f(x)) - x}{\alpha} = \nabla_{\Omega} f(x). \quad \square$$

This result is a consequence of Lemma 4.6 in Zarantonello (1971). Also see Proposition 2 in McCormick and Tapia (1972).

### 3. Trust region methods

In this section we present a trust region method for the solution of problem (1.2) with a closed convex set  $\Omega$ . This algorithm was proposed by Moré (1988) as a modification of the algorithm of Toint (1988). The development in this section follows Moré (1988).

At each iteration of a trust region method there is an approximation  $x_k \in \Omega$  to the solution, a bound  $\Delta_k$ , and a model  $\psi_k : \mathbb{R}^n \rightarrow \mathbb{R}$  of the possible reduction  $f(x_k + w) - f(x_k)$  for  $\|w\| \leq \Delta_k$ . We assume that  $\psi_k$  is defined if  $x_k + w \in \Omega$  and that

$$\psi_k(0) = 0, \quad \nabla \psi_k(0) = \nabla f(x_k).$$

We follow Toint (1988) in allowing a general  $\psi_k$ . An interesting choice of  $\psi_k$  is the nonlinear model defined by

$$\psi_k(w) = f(x_k + w) - f(x_k). \tag{3.1}$$

In many situations the model  $\psi_k$  is a quadratic, and thus

$$\psi_k(w) = \langle \nabla f(x_k), w \rangle + \frac{1}{2} \langle w, B_k w \rangle, \tag{3.2}$$

for some symmetric matrix  $B_k$ . Of course, it is possible to choose  $B_k = 0$ , and then the model is linear.

The iterate  $x_k$  and the bound  $\Delta_k$  are updated according to rules that are standard in trust region methods for unconstrained minimization. See, for example, Moré (1983). Given a step  $s_k$  such that  $x_k + s_k \in \Omega$  and  $\psi_k(s_k) < 0$ , these rules depend on the ratio

$$\rho_k = \frac{f(x_k + s_k) - f(x_k)}{\psi_k(s_k)}$$

of the actual reduction in the function to the predicted reduction in the model. Since the step  $s_k$  is chosen so that  $\psi_k(s_k) < 0$ , a step with  $\rho_k > 0$  yields a reduction in the function. Given  $\eta_1 > 0$ , the iterate  $x_k$  is updated as follows:

If  $\rho_k > \eta_1$  then  $x_{k+1} = x_k + s_k$ .

If  $\rho_k \leq \eta_1$  then  $x_{k+1} = x_k$ .

The iterates with  $\rho_k > \eta_1$  play an important role in the convergence analysis and thus we define

$$\mathcal{S} = \{k: \rho_k > \eta_1\}$$

as the sequence of *successful* iterations. The updating rules for  $\Delta_k$  depend on a constant  $\eta_2$  such that

$$0 < \eta_1 < \eta_2 < 1,$$

while the rate at which  $\Delta_k$  is either increased or decreased depend on constants  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  such that

$$0 < \sigma_1 < \sigma_2 < 1 < \sigma_3.$$

The trust region bound  $\Delta_k$  is updated as follows:

If  $\rho_k \leq \eta_1$  then  $\Delta_{k+1} \in [\sigma_1 \Delta_k, \sigma_2 \Delta_k]$ .

If  $\rho_k \in (\eta_1, \eta_2)$  then  $\Delta_{k+1} \in [\sigma_1 \Delta_k, \sigma_3 \Delta_k]$ .

If  $\rho_k \geq \eta_2$  then  $\Delta_{k+1} \in [\Delta_k, \sigma_3 \Delta_k]$ .

Variations on this updating scheme have been used, for example, by Moré (1983), Conn, Gould and Toint (1988a), Toint (1988) and Moré (1988). However, these authors assume that  $\Delta_{k+1} \leq \Delta_k$  if  $\rho_k \in (\eta_1, \eta_2)$ . In some situations this is an undesirable restriction. Another variation is to allow

$$\Delta_{k+1} \in [\sigma_1 \|s_k\|, \sigma_1 \Delta_k]$$

whenever  $\rho_k \leq \eta_2$ . As we shall see, our analysis also applies to this variation.

We follow the suggestion of Toint (1988) and choose a step  $s_k$  that gives as much reduction in the model  $\psi_k$  as one step of the gradient projection method applied to the subproblem

$$\min\{\psi_k(w): x_k + w \in \Omega, \|w\| \leq \Delta_k\}. \quad (3.3)$$

The step generated by the gradient projection algorithm is of the form  $s_k(\alpha_k)$  where the function  $s_k(\cdot)$  is defined by

$$s_k(\alpha) = P(x_k - \alpha \nabla f(x_k)) - x_k \quad (3.4)$$

and  $\alpha_k$  satisfies the following two requirements. Given constants  $\mu_0, \mu_1$  and  $\mu_2$  such that

$$0 < \mu_0 < \frac{1}{2}, \quad 0 < \mu_1 < \mu_2,$$

the first requirement is that

$$\psi_k(s_k(\alpha_k)) \leq \mu_0 \langle \nabla f(x_k), s_k(\alpha_k) \rangle \quad \text{and} \quad \|s_k(\alpha_k)\| \leq \mu_2 \Delta_k, \tag{3.5}$$

while the second requirement is that there are positive constants  $\gamma_1$  and  $\gamma_2$  such that

$$\alpha_k \geq \gamma_1 \quad \text{or} \quad \alpha_k \geq \gamma_2 \bar{\alpha}_k, \tag{3.6}$$

where  $\bar{\alpha}_k > 0$  satisfies

$$\psi_k(s_k(\bar{\alpha}_k)) \geq (1 - \mu_0) \langle \nabla f(x_k), s_k(\bar{\alpha}_k) \rangle \quad \text{or} \quad \|s_k(\bar{\alpha}_k)\| \geq \mu_1 \Delta_k. \tag{3.7}$$

The first requirement on  $\psi_k$  guarantees that the step  $s_k(\alpha_k)$  produces a sufficient reduction, while the second requirement guarantees that the step is not too small. These requirements are illustrated in Fig. 1 for a quadratic  $\psi_k$  and a polyhedral  $\Omega$ . Since  $\Omega$  is polyhedral,  $\langle \nabla f(x_k), s_k(\cdot) \rangle$  is a piecewise linear function and  $\psi_k(s_k(\cdot))$  is a piecewise quadratic function. Moreover, both functions are constant after the last breakpoint; in this case  $\alpha = 1.5$  is the last breakpoint. In Fig. 1 the set of  $\alpha_k$  which satisfy the sufficient decrease condition

$$\psi_k(s_k(\alpha_k)) \leq \mu_0 \langle \nabla f(x_k), s_k(\alpha_k) \rangle$$

consists of two disjoint intervals; approximately  $[0, 0.8]$  and  $[1.3, \infty)$ . The set of  $\bar{\alpha}_k$  which satisfy the condition

$$\psi_k(s_k(\bar{\alpha}_k)) \geq (1 - \mu_0) \langle \nabla f(x_k), s_k(\bar{\alpha}_k) \rangle$$

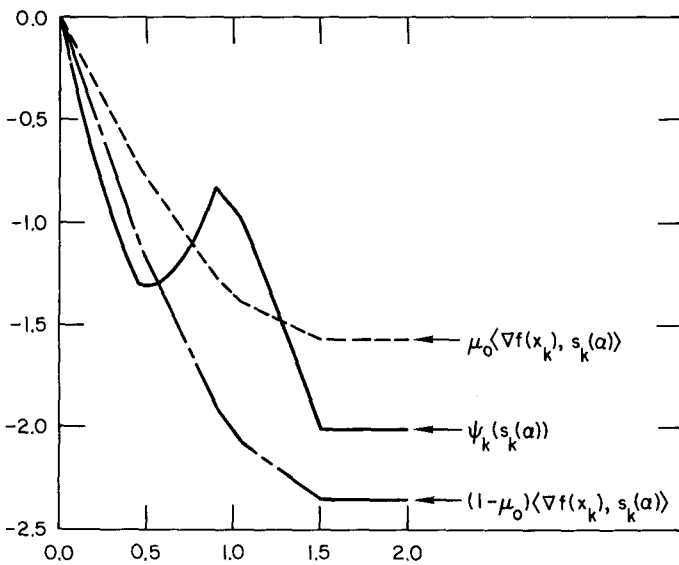


Fig. 1. Acceptance criteria for  $\alpha_k$ .

is approximately the interval  $[0.6, \infty)$ . Thus, for example, the set of  $\alpha_k$  which satisfy both conditions with  $\alpha_k \geq \gamma_2 \bar{\alpha}_k$  and  $\gamma_2 = \frac{1}{3}$  is roughly the union of the intervals  $[0.2, 0.8]$  and  $[1.3, \infty)$ . This set of acceptable steps may be further restricted by the trust region constraint, but this restriction is not considered in Fig. 1.

Moré (1988) proved that (3.5) and (3.6) can be satisfied with a finite number of evaluations of  $s_k(\cdot)$ . In practice we expect that only one or two evaluations of  $s_k(\cdot)$  will be needed to obtain an appropriate  $\alpha_k$ . We also note that for special constraint sets  $\Omega$ , the projection can be evaluated efficiently. This is the case, for example, if  $\Omega$  is a simplex, or if  $\Omega$  is defined by bound constraints. For a general polyhedral  $\Omega$ , the evaluation of  $s_k(\cdot)$  requires the solution of a least distance problem.

Motivation for requirements (3.5) and (3.6) on  $\alpha_k$  can be found in the work of Calamai and Moré (1987) on convergence properties of the gradient projection method. Indeed, Toint (1988) pointed out that if the model is defined by (3.1) and if the step is not restricted by the trust region bound, then (3.5) and (3.6) are the requirements imposed on  $\alpha_k$  by Calamai and Moré (1987).

The step  $s_k$  is required to satisfy a requirement similar to (3.5). We follow Toint (1988) in assuming that the step  $s_k$  satisfies

$$\psi_k(s_k) \leq \mu_0 \psi_k(s_k(\alpha_k)), \quad \|s_k\| \leq \mu_2 \Delta_k, \quad x_k + s_k \in \Omega. \tag{3.8}$$

In particular, this allows the choice of  $s_k = s_k(\alpha_k)$ . It is possible to weaken (3.8) further because in Section 4 we show that our convergence results hold if (3.8) is replaced by the requirement that there is a constant  $\mu_3 > 0$  with

$$-\psi_k(s_k) \geq \mu_0^2 \mu_3 \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \min \left\{ \Delta_k, \frac{1}{\beta_k} \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \right\}, \tag{3.9}$$

where  $\hat{\alpha}_k = \min\{\alpha_k, \gamma_3\}$  for any constant  $\gamma_3 > 0$ .

When  $\psi_k$  is a quadratic and  $\Omega$  is  $\mathbb{R}^n$ , it is standard to choose  $\alpha_k$  as a global minimizer of  $\psi_k(s_k(\alpha))$  subject to the condition that  $\|s_k(\alpha)\| \leq \mu_2 \Delta_k$ . For this choice (3.5) and (3.6) hold with  $\bar{\alpha}_k = \alpha_k$ . Also note that in this case (3.8) and (3.9) reduce to standard conditions for unconstrained problems. See, for example, Moré (1983) and Schultz, Schnabel and Byrd (1985).

When  $\psi_k$  is a quadratic and  $\Omega$  is defined by bound constraints, Conn, Gould and Toint (1988a) choose  $\alpha_k$  as the first local minimizer of  $\psi_k(s_k(\alpha))$  subject to the condition that  $\|s_k(\alpha)\| \leq \mu_2 \Delta_k$ . This choice does not necessarily satisfy conditions (3.5) and (3.6), and is therefore not covered by our theory. Also note that this choice of  $\alpha_k$  may not be defined for a non-polyhedral  $\Omega$  because the function  $\psi_k(s_k(\alpha))$  can be strictly decreasing for all  $\alpha > 0$ .

We assume that if  $x_k$  is a stationary point for problem (1.2) then the trust region method terminates at  $x_k$ . If  $x_k$  is not a stationary point then (2.6) implies that  $\langle \nabla f(x_k), s_k(\alpha) \rangle < 0$  for any  $\alpha > 0$ , and thus  $\psi_k(s_k) < 0$  for any step  $s_k$  which satisfies (3.5) and (3.8). In particular, it is possible to compute  $x_{k+1}$ .

We can introduce a scaling matrix  $D_k$  in the trust region method by replacing the norm  $\|\cdot\|$  by a scaled norm  $\|D_k(\cdot)\|$  in the requirements on  $\alpha_k$  and  $s_k$ . If the

matrices  $D_k$  are nonsingular with  $D_k$  and  $D_k^{-1}$  uniformly bounded, then the unscaled requirements are satisfied with different values of  $\mu_1$  and  $\mu_2$ . A similar argument based on the equivalence of norms in  $\mathbb{R}^n$  shows that there is no need to require that  $\|\cdot\|$  be an inner product norm in the requirements on  $\alpha_k$  and  $s_k$ .

**4. Basic convergence theory**

The first step in the convergence analysis of the trust region method of Section 3 is to obtain an estimate on the predicted decrease by the gradient projection step. This estimate is expressed in terms of the function  $\omega_k : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\omega_k(s) = \frac{\psi_k(s) - \psi_k(0) - \langle \nabla \psi_k(0), s \rangle}{\|s\|^2}.$$

The convergence properties of the trust region method depend on the properties of  $\omega_k(\cdot)$  for  $s \neq 0$ . Note that it is not unreasonable to assume that  $\omega_k(\cdot)$  is uniformly bounded. This happens, for example, if  $\psi_k$  is the quadratic (3.2) and the matrices  $\{B_k\}$  are uniformly bounded, or if  $\psi_k$  is defined by (3.1) and  $\nabla f$  is Lipschitz continuous. However, in this section we are interested in the case where  $\omega_k(\cdot)$  may fail to be uniformly bounded.

**Theorem 4.1.** *Define the bound*

$$\beta_k = 1 + \sup\{|\omega_k(s)| : 0 < \|s\| \leq \mu_2 \Delta_k\},$$

and for any constant  $\gamma_3 > 0$  let

$$\hat{\alpha}_k = \min\{\alpha_k, \gamma_3\}.$$

If  $\alpha_k$  satisfies (3.5) and (3.6) then there is a constant  $\mu_3 > 0$  such that

$$-\langle \nabla f(x_k), s_k(\alpha_k) \rangle \geq \mu_3 \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \min \left\{ \Delta_k, \frac{1}{\beta_k} \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \right\}. \quad \square$$

Theorem 4.1 was obtained by Moré (1988) as an improvement on the estimate of Toint (1988). An immediate consequence of Theorem 4.1 and assumptions (3.5) and (3.8) is that

$$-\psi_k(s_k) \geq \mu_0^2 \mu_3 \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \min \left\{ \Delta_k, \frac{1}{\beta_k} \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \right\}. \tag{4.1}$$

In this section we prove that our convergence results hold if the requirement (3.8) on  $\psi_k(s_k)$  is replaced by (4.1). This bound on the predicted decrease of  $\psi_k$  shows that since  $\rho_k > \eta_1$  for successful iterations,

$$f(x_k) - f(x_{k+1}) \geq \mu_4 \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \min \left\{ \Delta_k, \frac{1}{\beta_k} \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \right\}, \tag{4.2}$$

where  $\mu_4 = \eta_1 \mu_0^2 \mu_3$ . These two estimates will be used in our convergence analysis.

The convergence properties of the trust region method depend on the behavior of the sequence  $\{\beta_k\}$ . The following convergence result was obtained by Moré (1988) under the assumption that  $f$  is bounded below on  $\Omega$ . However, inspection of the proof shows that the result holds under the assumption that  $\{f(x_k)\}$  is bounded below.

**Theorem 4.2.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$  and assume that  $\{f(x_k)\}$  is bounded below. If  $\{\beta_k\}$  is uniformly bounded then*

$$\liminf_{k \rightarrow \infty} \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} = 0. \quad \square$$

The assumption that  $\{\beta_k\}$  is uniformly bounded holds for several important choices of the model  $\psi_k$ , but it does not apply, for example, to the *hypercube* method of Fletcher (1972) where  $\psi_k$  is the quadratic (3.2) and the matrices  $\{B_k\}$  are generated by a quasi-Newton update. The following result is patterned after a similar result in Powell (1984) and Toint (1988). Note, however, that our argument seems to be more direct.

**Lemma 4.3.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$ , and assume that  $\{f(x_k)\}$  is bounded below and that  $\nabla f$  is uniformly continuous on  $\Omega$ . If*

$$\|s_k(\hat{\alpha}_k)\| / \hat{\alpha}_k \geq \epsilon \tag{4.3}$$

for some  $\epsilon > 0$  and all  $k \geq 0$ , then there is an  $\epsilon_0 > 0$  such that

$$\hat{\beta}_k \Delta_k \geq \epsilon_0, \quad k \geq 0, \tag{4.4}$$

where

$$\hat{\beta}_k = \max\{\beta_j : 0 \leq j \leq k\}.$$

**Proof.** We first show that there is an  $\epsilon_1 > 0$  such that if  $\rho_k < \eta_2$  then

$$\beta_k \|s_k\| \geq \epsilon_1 \tag{4.5}$$

for all  $k$  sufficiently large. Assume, on the contrary, that there is an infinite subsequence  $\mathcal{H}$  such that  $\rho_k < \eta_2$  and  $\{\beta_k \|s_k\|\}$  converges to zero for  $k \in \mathcal{H}$ . Since  $\beta_k \geq 1$ , this implies that  $\{\|s_k\|\}$  converges to zero for  $k \in \mathcal{H}$ , and since  $\nabla f$  is uniformly continuous, there is a sequence  $\{\epsilon_k\}$  converging to zero such that

$$|f(x_k + s_k) - f(x_k) - \langle \nabla f(x_k), s_k \rangle| \leq \epsilon_k \|s_k\|.$$

Now note that since  $\psi_k(0) = 0$  and  $\nabla \psi_k(0) = \nabla f(x_k)$ ,

$$\begin{aligned} &|f(x_k + s_k) - f(x_k) - \psi_k(s_k)| \\ &\leq |f(x_k + s_k) - f(x_k) - \langle \nabla f(x_k), s_k \rangle| + |\psi_k(s_k) - \psi_k(0) - \langle \psi_k(0), s_k \rangle|, \end{aligned}$$

and thus the definition of  $\beta_k$  implies that

$$|f(x_k + s_k) - f(x_k) - \psi_k(s_k)| \leq \epsilon_k \|s_k\| + \beta_k \|s_k\|^2.$$

Another estimate is needed in order to obtain a contradiction. Note that  $\{\beta_k \|s_k\|\}$  converges to zero, and thus  $\|s_k\| \leq \mu_2 \Delta_k$ , and the bound (4.1) imply that

$$-\psi_k(s_k) \geq \mu_0^2 \mu_3 \epsilon \|s_k\|$$

for all  $k \in \mathcal{K}$ . The last two estimates yield that  $\{\rho_k - 1\}$  converges to zero for  $k \in \mathcal{K}$ . This contradiction establishes that (4.5) holds for all  $k$  sufficiently large if  $\rho_k < \eta_2$ . If necessary, restrict  $\epsilon_1$  further so that (4.5) holds for all  $k \geq 0$  with  $\rho_k < \eta_2$ . We now use an induction argument to prove that (4.4) holds for  $\epsilon_0 = \sigma_1 \epsilon_1$  and all  $k \geq 0$ . If (4.4) holds for some  $k > 0$  and  $\rho_k \geq \eta_2$  then (4.4) holds for  $k + 1$  because  $\hat{\beta}_{k+1} \geq \hat{\beta}_k$  and  $\Delta_{k+1} \geq \Delta_k$ . If  $\rho_k < \eta_2$  then the updating rules for  $\Delta_k$  and (4.5) imply that

$$\beta_k \Delta_{k+1} \geq \beta_k \sigma_1 \|s_k\| \geq \sigma_1 \epsilon_1,$$

and thus (4.4) holds for  $k + 1$  because  $\hat{\beta}_{k+1} \geq \beta_k$ .  $\square$

We now present the main convergence result of this section. We weaken the assumptions of Theorem 4.2 by assuming that the series

$$\sum_{k=1}^{\infty} \frac{1}{\hat{\beta}_k} \tag{4.6}$$

is divergent. On the other hand, we now need to assume that  $\nabla f$  is uniformly continuous.

**Theorem 4.4.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$ , and assume that  $\{f(x_k)\}$  is bounded below and that  $\nabla f$  is uniformly continuous on  $\Omega$ . If the series (4.6) is divergent then*

$$\liminf_{k \rightarrow \infty} \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} = 0.$$

**Proof.** We prove this result by showing that if (4.3) holds then (4.6) is convergent. The proof of this result can be split in two parts. For either part it is necessary to consider the set

$$\mathcal{P} = \{k: q(k) \geq k/p\},$$

where  $q(k)$  is the number of successful iterations whose index does not exceed  $k$ , and  $p > 1$  is chosen such that  $\sigma_3 \sigma_2^{p-1} < 1$ . In the first part we prove that

$$\sum_{k \notin \mathcal{P}} \frac{1}{\hat{\beta}_k} \tag{4.7}$$

is convergent, while in the second part we show that

$$\sum_{k \in \mathcal{P}} \frac{1}{\hat{\beta}_k} \tag{4.8}$$

is convergent. The convergence of (4.7) and (4.8) imply the convergence of (4.6), which is the desired contradiction.

The proof of the first part uses an induction argument to show that

$$\Delta_k \leq \sigma_3^{q(k)} \sigma_2^{k-q(k)} \Delta_0$$

for any  $k \geq 0$ , and thus

$$\Delta_k \leq [\sigma_3 \sigma_2^{p-1}]^{k/p} \Delta_0$$

for  $k \notin \mathcal{P}$ . Since Lemma 4.3 shows that  $\epsilon_0/\hat{\beta}_k \leq \Delta_k$ , and since  $\sigma_3 \sigma_2^{p-1} < 1$ , the convergence of (4.7) is established.

For the proof of the second part we first need to consider the sequence  $\{k_i\}$  of successful iterations and prove that

$$\sum_{i=1}^{\infty} \frac{1}{\hat{\beta}_{k_i}} \tag{4.9}$$

is convergent. This is established by noting that estimate (4.2) implies that

$$f(x_k) - f(x_{k+1}) \geq \mu_4 \epsilon \min\{\Delta_k, \epsilon/\beta_k\}$$

for all successful iterations, and thus Lemma 4.3 shows that there is an  $\epsilon_1 > 0$  such that

$$f(x_k) - f(x_{k+1}) \geq \epsilon_1/\hat{\beta}_k$$

for all successful iterations. This estimate shows that (4.9) is convergent. Hence, (4.8) is convergent if we prove that

$$\sum_{k \in \mathcal{P}} \frac{1}{\hat{\beta}_k} \leq p \sum_{i=1}^{\infty} \frac{1}{\hat{\beta}_{k_i}}. \tag{4.10}$$

We claim that if  $\mathcal{P}_i = \mathcal{P} \cap [ip, (i+1)p - 1]$  then  $\hat{\beta}_k \geq \hat{\beta}_{k_i}$  for  $k \in \mathcal{P}_i$ . This holds if  $k_i \leq ip$  because  $\{\hat{\beta}_k\}$  is nondecreasing. If  $k_i > ip$  and  $ip \leq k < k_i$  then  $q(k) < i \leq k/p$ , and thus  $k \notin \mathcal{P}$ . Since  $\hat{\beta}_k \geq \hat{\beta}_{k_i}$  for  $k \in \mathcal{P}_i$ ,

$$\sum_{k \in \mathcal{P}_i} \frac{1}{\hat{\beta}_k} \leq \frac{p}{\hat{\beta}_{k_i}},$$

and thus it is clear that (4.10) holds. Since we have already shown the convergence of (4.9), this implies the convergence of (4.8), and thus completes the proof of the second part.  $\square$

The argument used in the proof of Theorem 4.4 is similar to the argument used by Powell (1984) and Toint (1988); the main difference occurs in the proof of inequality (4.10). Also note that the assumptions of Theorem 4.4 are weaker than those in the comparable result of Toint (1988) which assumes that  $\nabla f$  is Lipschitz continuous on  $\Omega$  and that  $\Omega$  is bounded. Moreover, Toint (1988) proves that

$$\liminf_{k \rightarrow \infty} \|s_k(1)\| = 0.$$



This result follows from Theorem 4.4 because Lemma 2.4 and (2.8) imply that

$$\frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \geq \frac{\|s_k(\gamma_3)\|}{\gamma_3} \geq \frac{\min\{\gamma_3, 1\}}{\gamma_3} \|s_k(1)\|.$$

As we shall see in the next section, the stronger result of Theorem 4.4 is of importance.

**Theorem 4.5.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$ , and assume that  $\{f(x_k)\}$  is bounded below and that  $\nabla f$  is uniformly continuous on  $\Omega$ . If the series (4.6) is divergent then the sequence  $\mathcal{S}$  of successful iterations is infinite.*

**Proof.** If all iterates  $k \geq k_0$  are unsuccessful then  $s_k(\alpha) = s_{k_0}(\alpha)$  for all  $\alpha > 0$  and  $k \geq k_0$ . Since  $\hat{\alpha}_k \leq \gamma_3$ , Lemma 2.4 implies that

$$\frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \geq \frac{\|s_k(\gamma_3)\|}{\gamma_3} = \frac{\|s_{k_0}(\gamma_3)\|}{\gamma_3}$$

for  $k \geq k_0$ . Theorem 4.4 shows that  $s_{k_0}(\gamma_3) = 0$ , and thus (2.7) yields that  $x_{k_0}$  is a stationary point for problem (1.2). This contradicts our assumption that no iterate is a stationary point.  $\square$

### 5. Cauchy points

An important aspect of the convergence analysis is the relationship between the trust region method and the projected gradient. The main result of this section shows that we can define a related sequence  $\{x_k^C\}$  such that if  $x^*$  is a limit point of  $\{x_k\}$  then there is a subsequence  $\{x_{k_i}\}$  which converges to  $x^*$  with

$$\lim_{i \rightarrow \infty} \|\nabla_{\Omega} f(x_{k_i}^C)\| = 0.$$

Moreover,  $\{x_{k_i}^C\}$  also converges to  $x^*$ . An important consequence of this result is that every limit point of  $\{x_k\}$  is a stationary point.

The *Cauchy point*  $x_k^C$  is defined in terms of the gradient projection method. For any constant  $\gamma_3 > 0$ , set  $\hat{\alpha}_k = \min\{\alpha_k, \gamma_3\}$  as in Theorem 4.1, and define

$$x_k^C = P(x_k - \hat{\alpha}_k \nabla f(x_k)) = x_k + s_k(\hat{\alpha}_k).$$

Note that  $x_k^C$  belongs to  $\Omega$  and that  $\|x_k^C - x_k\| \leq \mu_2 \Delta_k$ . The following technical result relates the projected gradient at the Cauchy point to the results of Section 4.

**Lemma 5.1.** *If  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable on  $\Omega$  and  $x_k \in \Omega$  then*

$$\|\nabla_{\Omega} f(x_k^C)\| \leq \|\nabla f(x_k^C) - \nabla f(x_k)\| + \|x_k^C - x_k\| / \hat{\alpha}_k.$$

**Proof.** The basic inequality (2.6) implies that

$$\hat{\alpha}_k \langle \nabla f(x_k), x_k^C - z_k \rangle \leq -\langle x_k^C - x_k, x_k^C - z_k \rangle \leq \|x_k^C - x_k\| \|x_k^C - z_k\|$$

for all  $z_k \in \Omega$ . Hence, if  $v_k$  is a feasible descent direction at  $x_k^C$  with  $\|v_k\| \leq 1$ , then  $x_k^C + \tau_k v_k$  belongs to  $\Omega$  for some  $\tau_k > 0$ , and thus setting  $z_k = x_k^C + \tau_k v_k$  yields

$$-\langle \nabla f(x_k), v_k \rangle \leq \|x_k^C - x_k\| / \hat{\alpha}_k.$$

Hence

$$-\langle \nabla f(x_k^C), v_k \rangle \leq \|\nabla f(x_k^C) - \nabla f(x_k)\| + \|x_k^C - x_k\| / \hat{\alpha}_k.$$

Part (c) of Lemma 2.1 yields the desired result.  $\square$

A consequence of Lemma 5.1 is that the sequence  $\{\|\nabla_{\Omega} f(x_k^C)\|\}$  is not bounded away from zero. The proof of this result follows from Theorem 4.4 by noting that since  $\{\hat{\alpha}_k\}$  is bounded above,  $\{x_k^C - x_k\}$  converges to zero. This establishes the following result.

**Theorem 5.2.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$ , and assume that  $\{f(x_k)\}$  is bounded below and that  $\nabla f$  is uniformly continuous on  $\Omega$ . If the series (4.6) is divergent then*

$$\liminf_{k \rightarrow \infty} \|\nabla_{\Omega} f(x_k^C)\| = 0. \quad \square$$

Stronger results than Theorem 5.2 need further assumptions on the sequence  $\{\beta_k\}$ . Toint (1988) assumes that the series (4.6) is divergent and that

$$\lim_{k \rightarrow \infty} \beta_k [f(x_k) - f(x_{k+1})] = 0. \quad (5.1)$$

An unsatisfactory aspect of this assumption is that it cannot be verified *a priori*; we prefer to assume that the sequence  $\{\beta_k\}$  is bounded. However, note that our results hold if we assume (5.1) and that (4.6) is divergent.

A weakness of Theorem 5.2 is that it does not provide information on the behavior of  $\|\nabla_{\Omega} f(x_k^C)\|$  when  $k$  is a successful iteration. The following result assumes that the sequence of models  $\{\psi_k\}$  is chosen so that convergence of

$$\sum_{k=1}^{\infty} \|s_k\| \quad (5.2)$$

implies that

$$\lim_{k \rightarrow \infty} \frac{f(x_k + s_k) - f(x_k) - \psi_k(s_k)}{\|s_k\|^2} = 0. \quad (5.3)$$

If  $\psi_k$  is the quadratic (3.2) then this assumption is certainly satisfied if  $B_k = \nabla^2 f(x_k)$ . More generally, if  $\psi_k$  is the quadratic (3.2) and  $f$  is twice continuously differentiable, then the continuity of  $\nabla^2 f$  implies that

$$f(x_k + x_k) - f(x_k) - \langle \nabla f(x_k), s_k \rangle = \frac{1}{2} \langle s_k, \nabla^2 f(x_k + \tau_k s_k) s_k \rangle$$

for some  $\tau_k \in [0, 1]$ , and thus (5.3) is equivalent to

$$\lim_{k \rightarrow \infty} \frac{\langle s_k, [B_k - \nabla^2 f(x_k)] s_k \rangle}{\|s_k\|^2} = 0,$$

whenever  $\{x_k\}$  converges and  $\{s_k\}$  converges to zero. As a final example, note that (5.3) holds automatically for the nonlinear model  $\psi_k(w) = f(x_k + w) - f(x_k)$ .

We also assume that the sequence  $\{\beta_k\}$  is bounded; for the quadratic model (3.2) this is equivalent to assuming that the sequence  $\{B_k\}$  is bounded.

**Theorem 5.3.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$  and assume that  $\{f(x_k)\}$  is bounded below. If  $\{\beta_k\}$  is bounded and if (5.3) holds whenever (5.2) is convergent, then*

$$\liminf_{k \in \mathcal{S}} \|\nabla_{\Omega} f(x_k^c)\| = 0.$$

**Proof.** Lemma 5.1 shows that we only need to prove that

$$\liminf_{k \in \mathcal{S}} \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} = 0.$$

We first outline the proof and then present the details. The proof is by contradiction. We assume that there is an  $\epsilon > 0$  such that

$$\|s_k(\hat{\alpha}_k)\| / \hat{\alpha}_k \geq \epsilon, \quad k \in \mathcal{S}. \tag{5.4}$$

The main part of the proof consists of showing that assumption (5.4) implies that

$$\liminf_{k \in \mathcal{S}} \frac{\|s_k(\hat{\alpha}_k)\|}{\Delta_k} > 0, \tag{5.5}$$

$$\lim_{k \notin \mathcal{S}} \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} = 0, \tag{5.6}$$

$$\lim_{k \notin \mathcal{S}} \frac{\|s_k(\hat{\alpha}_k)\|}{\|s_k\|} = 0. \tag{5.7}$$

We now show how these three results lead to a contradiction.

Consider an infinite sequence  $\mathcal{H}$  such that  $\mathcal{S} \cap \mathcal{H}$  is empty, and such that  $k \in \mathcal{H}$  implies that  $k + 1$  belongs to  $\mathcal{S}$ . Since all iterations in  $\mathcal{H}$  are unsuccessful,  $s_{k+1}(\alpha) = s_k(\alpha)$  for all  $\alpha > 0$ . Thus, if  $\hat{\alpha}_{k+1} \geq \hat{\alpha}_k$  for infinitely many  $k \in \mathcal{H}$ , then Lemma 2.4 and (5.4) imply that

$$\epsilon \leq \frac{\|s_{k+1}(\hat{\alpha}_{k+1})\|}{\hat{\alpha}_{k+1}} \leq \frac{\|s_{k+1}(\hat{\alpha}_k)\|}{\hat{\alpha}_k} = \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k}, \quad k \in \mathcal{H}.$$

In view of (5.6), this cannot happen infinitely often. We have thus shown that  $\hat{\alpha}_{k+1} \leq \hat{\alpha}_k$  for all  $k \in \mathcal{H}$  sufficiently large, and hence Lemma 2.4 shows that

$$\|s_{k+1}(\hat{\alpha}_{k+1})\| \leq \|s_{k+1}(\hat{\alpha}_k)\| = \|s_k(\hat{\alpha}_k)\|, \quad k \in \mathcal{H}. \tag{5.8}$$

Since the updating rules for  $\Delta_k$  imply that  $\sigma_1 \|s_k\| \leq \Delta_{k+1}$  for any index  $k$ , the limit (5.7) and inequality (5.8) show that

$$\limsup_{k \in \mathcal{H}} \frac{\|s_{k+1}(\hat{\alpha}_{k+1})\|}{\Delta_{k+1}} \leq 0.$$

This contradicts (5.5) and establishes the result. We complete the proof by showing that assumption (5.4) implies that (5.5), (5.6) and (5.7) hold.

As a preliminary step, we show that assumption (5.4) implies that the series (5.2) is convergent. Note that the bound (4.2) on the actual reduction for a successful iteration yields

$$f(x_k) - f(x_{k+1}) \geq \mu_4 \epsilon \min\{\Delta_k, \epsilon/\beta_k\}, \quad k \in \mathcal{S}.$$

Since  $\{f(x_k)\}$  is bounded below and  $\{\beta_k\}$  is bounded,  $\{\Delta_k\}$  converges to zero for  $k \in \mathcal{S}$ . In particular, this implies that  $\Delta_k \leq \epsilon/\beta_k$ , and hence

$$f(x_k) - f(x_{k+1}) \geq \mu_4 \epsilon \Delta_k, \quad k \in \mathcal{S}.$$

Thus, since  $\{f(x_k)\}$  is bounded below,

$$\sum_{k \in \mathcal{S}} \Delta_k$$

is a convergent series. A careful analysis of the updating rules for  $\Delta_k$  show that convergence over the successful iterates implies that the full series

$$\sum_{k=1}^{\infty} \Delta_k$$

is convergent. Hence, (5.2) is convergent, and by assumption, (5.3) holds.

We now show that assumption (5.4) implies that (5.5) holds. Since  $\|s_k(\hat{\alpha}_k)\| \leq \mu_2 \Delta_k$  and  $\{\Delta_k\}$  converges to zero, assumption (5.4) implies that  $\{\hat{\alpha}_k\}$  converges to zero for  $k \in \mathcal{S}$ . Hence, we eventually have  $\hat{\alpha}_k = \alpha_k \geq \gamma_2 \bar{\alpha}_k$  where  $\bar{\alpha}_k$  satisfies (3.7). Assume that  $\bar{\alpha}_k$  satisfies the first condition in (3.7). The characterization (2.6) of the projection implies that

$$-\langle \nabla f(x_k), s_k(\alpha) \rangle \geq \|s_k(\alpha)\|^2 / \alpha$$

for all  $\alpha > 0$ , and thus

$$\beta_k > \omega_k(s_k(\bar{\alpha}_k)) > -\mu_0 \frac{\langle \nabla f(x_k), s_k(\bar{\alpha}_k) \rangle}{\|s_k(\bar{\alpha}_k)\|^2} \geq \frac{\mu_0}{\bar{\alpha}_k}.$$

Since  $\{\beta_k\}$  is bounded and  $\hat{\alpha}_k \geq \min\{\gamma_2 \bar{\alpha}_k, \gamma_3\}$ , this shows that  $\{\hat{\alpha}_k\}$  is bounded away from zero. This contradicts our earlier conclusion that  $\{\hat{\alpha}_k\}$  converges to zero for  $k \in \mathcal{S}$ . Hence, eventually all  $k \in \mathcal{S}$  satisfy the second condition in (3.7). Inequality (2.8) now implies that

$$\|s_k(\hat{\alpha}_k)\| = \|s_k(\alpha_k)\| \geq \min\{\gamma_2, 1\} \|s_k(\bar{\alpha}_k)\| \geq \min\{\gamma_2, 1\} \mu_1 \Delta_k,$$

and this show that (5.5) holds.

We now show that assumption (5.4) implies (5.6). Assume, on the contrary, that there is an infinite subset  $\mathcal{H}$  of unsuccessful iterations and an  $\epsilon_1 > 0$  such that

$$\|s_k(\hat{\alpha}_k)\| / \hat{\alpha}_k \geq \epsilon_1, \quad k \in \mathcal{H}.$$

The bound (4.1) and the assumption that the sequence  $\{\beta_k\}$  is bounded imply that there is an  $\epsilon_2 > 0$  such that

$$-\psi_k(s_k) \geq \epsilon_2 \Delta_k, \quad k \in \mathcal{K}.$$

Since  $\{x_k\}$  converges and  $\{s_k\}$  converges to zero, the continuity of  $\nabla f$  shows that there is a sequence  $\{\epsilon_k\}$  converging to zero such that

$$|f(x_k + s_k) - f(x_k) - \psi_k(s_k)| \leq \epsilon_k \|s_k\| + \beta_k \|s_k\|^2.$$

The last two estimates show that  $\{\rho_k - 1\}$  converges to zero for  $k \in \mathcal{K}$ . Hence, all iterations  $k \in \mathcal{K}$  are eventually successful. This contradiction establishes that (5.6) holds.

The proof that assumption (5.4) implies (5.7) is similar. Assume, on the contrary, that there is an infinite subset  $\mathcal{K}$  of unsuccessful iterations and an  $\epsilon_1 > 0$  such that

$$\|s_k(\hat{\alpha}_k)\| / \|s_k\| \geq \epsilon_1, \quad k \in \mathcal{K}.$$

The bound (4.1), the assumption that the sequence  $\{\beta_k\}$  is bounded, and the inequality  $\|s_k\| \leq \mu_2 \Delta_k$  imply that there is an  $\epsilon_2 > 0$  such that

$$-\psi_k(s_k) \geq \epsilon_2 \|s_k\|^2.$$

Since (5.3) holds,  $\{\rho_k - 1\}$  converges to zero for  $k \in \mathcal{K}$ . Hence, all iterations  $k \in \mathcal{K}$  are eventually successful. This contradiction establishes that (5.7) holds.  $\square$

Theorems 5.2 and 5.3 need to be developed further in order to obtain the main results of this section. The following result is the desired extension of Theorem 5.2.

**Theorem 5.4.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$  and assume that  $\{\beta_k\}$  is bounded. If  $x^*$  is a limit point of  $\{x_k\}$  then there is a subsequence  $\{x_{k_i}\}$  which converges to  $x^*$  with*

$$\lim_{i \rightarrow \infty} \|\nabla_{\Omega} f(x_{k_i}^C)\| = 0.$$

*Moreover,  $\{x_{k_i}^C\}$  also converges to  $x^*$ , and thus  $x^*$  is a stationary point for problem (1.2).*

**Proof.** Let  $\{x_{i_i}\}$  be any subsequence which converges to  $x^*$ . If

$$\lim_{i \rightarrow \infty} \frac{\|s_{i_i}(\hat{\alpha}_{i_i})\|}{\hat{\alpha}_{i_i}} = 0,$$

then Lemma 5.1 yields the result because  $s_k(\hat{\alpha}_k) = x_k^C - x_k$ . Assume that there is an  $\epsilon_0 > 0$  such that

$$\|s_{i_i}(\hat{\alpha}_{i_i})\| / \hat{\alpha}_{i_i} \geq \epsilon_0.$$

Theorem 4.2 guarantees that for any  $\epsilon$  in  $(0, \epsilon_0)$  there is a sequence  $\{m_i\}$  such that

$$\|s_k(\hat{\alpha}_k)\| / \hat{\alpha}_k \geq \epsilon, \quad l_i \leq k < m_i, \quad \|s_{m_i}(\hat{\alpha}_{m_i})\| / \hat{\alpha}_{m_i} < \epsilon.$$

Hence, the bound (4.2) on the actual decrease implies that if  $l_i \leq k < m_i$  then

$$f(x_k) - f(x_{k+1}) \geq \mu_4 \epsilon \min\{\Delta_k, \epsilon/\beta_k\}$$

for all successful iterations. Since  $\{\beta_k\}$  is bounded and  $\{f(x_k)\}$  converges,  $\{\Delta_k\}$  converges to zero for all successful iterations  $k$  such that  $l_i \leq k < m_i$ ; in particular,  $\Delta_k < \epsilon/\beta_k$ . Moreover, since  $\|s_k\| \leq \mu_2 \Delta_k$ , there is an  $\epsilon_1 > 0$  such that

$$f(x_k) - f(x_{k+1}) \geq \epsilon_1 \|x_{k+1} - x_k\|$$

for all successful iterations  $k$  with  $l_i \leq k < m_i$ . This inequality also holds if iteration  $k$  is not successful because  $x_{k+1} = x_k$  for unsuccessful iterations. Hence

$$f(x_{l_i}) - f(x_{m_i}) \geq \epsilon_1 \|x_{l_i} - x_{m_i}\|,$$

and thus  $\{x_{m_i}\}$  converges to  $x^*$ . In particular, we have shown that for any  $\epsilon$  in  $(0, \epsilon_0)$  there is an infinite sequence  $\mathcal{K}$  such that

$$\|s_k(\hat{\alpha}_k)\|/\hat{\alpha}_k \leq \epsilon, \quad \|x_k - x^*\| \leq \epsilon, \quad k \in \mathcal{K}.$$

This shows that there is a sequence  $\{k_i\}$  such that  $\{x_{k_i}\}$  converges to  $x^*$  and

$$\lim_{i \rightarrow \infty} \frac{\|s_{k_i}(\hat{\alpha}_{k_i})\|}{\hat{\alpha}_{k_i}} = 0.$$

Since  $s_k(\hat{\alpha}_k) = x_k^C - x_k$ , the application of Lemma 5.1 shows that  $\{\nabla_{\Omega} f(x_{k_i}^C)\}$  converges to zero. Moreover, since  $\{\hat{\alpha}_k\}$  is bounded above,  $\{x_{k_i}^C\}$  converges to  $x^*$ . Since Lemma 2.1 guarantees that the mapping  $\|\nabla_{\Omega} f(\cdot)\|$  is lower semicontinuous,  $\nabla_{\Omega} f(x^*) = 0$  and hence  $x^*$  is stationary.  $\square$

Note that Theorem 5.4 does not claim that the sequence  $\{k_i\}$  consists of successful iterations. A result along these lines can be obtained under the assumptions of Theorem 5.3 with a slight modification of the proof of Theorem 5.4. We shall see in Section 7 that this result is crucial to the development of the identification properties.

**Theorem 5.5.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$ , and assume that  $\{\beta_k\}$  is bounded and that (5.3) holds whenever (5.2) is convergent. If  $x^*$  is a limit point of  $\{x_k\}$  then there is a subsequence  $\{x_{k_i}\}$  of successful iterations which converges to  $x^*$  with*

$$\lim_{i \rightarrow \infty} \|\nabla_{\Omega} f(x_{k_i}^C)\| = 0.$$

Moreover,  $\{x_{k_i}^C\}$  also converges to  $x^*$ .

**Proof.** The proof follows that of Theorem 5.4 with only minor deviations. The only difference is that the proof starts with a sequence  $\{l_i\}$  of successful iterations such that  $\{x_{l_i}\}$  converges to  $x^*$ , and that Theorem 5.3 is used to guarantee that the sequence  $\{m_i\}$  can be chosen from the successful iterations.  $\square$

## 6. Convergence

A preliminary step in a local analysis of a trust region method is the development of conditions which guarantee that the iterates  $\{x_k\}$  converge. The results of this section show, in various contexts, that convergence is achieved if  $\{x_k\}$  has an *isolated* limit point  $x^*$ , that is, there is a neighborhood  $S(x^*)$  such that  $x^*$  is the only limit point in  $S(x^*) \cap \Omega$ .

We can guarantee that the sequence  $\{x_k\}$  generated by the trust region method of Section 3 has an isolated limit point  $x^*$  if we assume that  $x^*$  is a *strict minimizer* of  $f$ , that is, there is a neighborhood  $S(x^*)$  of  $x^*$  such that  $f(x^*) < f(x)$  for all  $x \neq x^*$  in  $S(x^*) \cap \Omega$ .

**Theorem 6.1.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous on  $\Omega$  and let  $\{x_k\}$  be an arbitrary sequence in  $\Omega$  such that  $f(x_{k+1}) \leq f(x_k)$  for  $k \geq 0$ . If  $\{x_k\}$  has a limit point  $x^*$  which is a strict minimizer of  $f$  in  $\Omega$  then  $x^*$  is an isolated limit point of  $\{x_k\}$ .*

**Proof.** Choose  $\epsilon > 0$  such that  $f(y) > f(x^*)$  whenever  $y \in \Omega$  and  $0 < \|y - x^*\| \leq \epsilon$ . If  $y^*$  is a limit point of  $\{x_k\}$  with  $\|y^* - x^*\| \leq \epsilon$  then the convergence of  $\{f(x_k)\}$  implies that  $f(y^*) = f(x^*)$  and thus  $y^* = x^*$ . This proves that  $x^*$  is an isolated limit point of  $\{x_k\}$ .  $\square$

The assumption that  $x^*$  is a strict minimizer can be guaranteed by imposing second order conditions on  $f$ . The following result uses a version of the second order sufficiency conditions that is appropriate for the general problem (1.2).

**Theorem 6.2.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$  and twice differentiable at a point  $x^*$  in  $\Omega$ . If  $x^*$  is a stationary point of problem (1.2) and*

$$\langle \nabla f(x^*), w \rangle = 0, \quad w \in T(x^*), \quad w \neq 0 \quad \Rightarrow \quad \langle w, \nabla^2 f(x^*) w \rangle > 0, \quad (6.1)$$

*then  $x^*$  is a strict minimizer of  $f$ .  $\square$*

Theorem 6.2 is a special case of a result of Robinson (1982, Theorem 2.4). In fact, Robinson shows that  $x^*$  is isolated in the sense that  $x^*$  is the only stationary point of problem (1.2) near  $x^*$ . Theorem 6.2 can also be derived as a special case of the results of Burke (1987).

For a polyhedral  $\Omega$  condition (6.1) coincides with the standard second order sufficiency conditions. For a general convex  $\Omega$  condition (6.1) has an advantage over the standard second order sufficiency conditions because it is independent of the representation of  $\Omega$ . On the other hand, the following example shows that condition (6.1) does not take into account the curvature of  $\Omega$ , and thus differs from the standard second order sufficiency conditions.

**Example.** Define  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $f(\xi_1, \xi_2) = \xi_2$ , and let

$$\Omega = \{(\xi_1, \xi_2): \xi_2 \geq \xi_1^2\}.$$

It is not difficult to verify that  $x^* = (0, 0)$  satisfies the standard second order conditions but that (6.1) fails when  $w = (1, 0)$ .

If  $x^*$  is a nondegenerate stationary point then (6.1) can be expressed in terms of  $N(x^*)^\perp$  where for any set  $S$  the *orthogonal complement*  $S^\perp$  of  $S$  is the subspace of vectors  $v$  such that  $\langle v, w \rangle = 0$  for all  $w \in S$ . The subspace  $N(x^*)^\perp$  is also known as the *lineality* of  $T(x^*)$  because it is the largest subspace in  $T(x^*)$ . Thus,

$$N(x^*)^\perp = T(x^*) \cap [-T(x^*)].$$

These notions are familiar when  $\Omega$  is the polyhedral set defined by (2.1) because then (2.3) shows that

$$N(x^*)^\perp = \{v \in \mathbb{R}^n : \langle c_j, v \rangle = 0, j \in \mathcal{A}(x^*)\}.$$

**Theorem 6.3.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$  and twice differentiable at a point  $x^*$  in  $\Omega$ . If  $x^*$  is a nondegenerate stationary point of problem (1.2) and*

$$w \in N(x^*)^\perp, w \neq 0 \implies \langle w, \nabla^2 f(x^*) w \rangle > 0, \quad (6.2)$$

*then  $x^*$  is a strict minimizer of  $f$ .*

**Proof.** The result follows from Theorem 6.2 if we prove that

$$N(x^*)^\perp = \{w : \langle \nabla f(x^*), w \rangle = 0, w \in T(x^*)\}.$$

A short computation shows that

$$N(x^*)^\perp \subset \{w : \langle \nabla f(x^*), w \rangle = 0, w \in T(x^*)\},$$

whenever  $x^*$  is a stationary point. The reverse inclusion holds if  $x^*$  is nondegenerate. We prove this by noting that since  $-\nabla f(x^*) \in \text{ri}(N(x^*))$  then

$$(1 - \lambda)(-\nabla f(x^*)) + \lambda v \in N(x^*)$$

for any  $v \in N(x^*)$  and  $|\lambda|$  sufficiently small. Hence, if  $\langle \nabla f(x^*), w \rangle = 0$  and  $w \in T(x^*)$  then  $\lambda \langle v, w \rangle \leq 0$  for  $|\lambda|$  sufficiently small. Since  $\lambda$  can be of either sign, this implies that  $\langle v, w \rangle = 0$ , and thus  $w \in N(x^*)^\perp$  as desired.  $\square$

Conditions (6.1) and (6.2) guarantee that  $x^*$  is a strict minimizer of  $f$ . Hence, under the assumptions of Theorem 6.1, these conditions show that  $x^*$  is an isolated limit point of  $\{x_k\}$ . The following result of Moré and Sorensen (1983) spells out the technical consequences of the assumption that  $x^*$  is an isolated limit point of  $\{x_k\}$ ; the proof is included for completeness.

**Lemma 6.4.** *Let  $\{x_k\}$  be an arbitrary sequence in  $\Omega$  and assume that  $x^*$  is an isolated limit point of  $\{x_k\}$ . Either  $\{x_k\}$  converges to  $x^*$ , or there is a sequence  $\{l_i\}$  such that  $\{x_{l_i}\}$  converges to  $x^*$  and an  $\epsilon > 0$  such that*

$$\|x_{l_i+1} - x_{l_i}\| \geq \epsilon, \quad i \geq 0.$$



**Proof.** Assume that  $\{x_k\}$  does not converge to  $x^*$  and choose  $\epsilon > 0$  such that if  $y^*$  is a limit point of  $\{x_k\}$  and  $\|y^* - x^*\| \leq \epsilon$  then  $y^* = x^*$ . If  $\|x_k - x^*\| \leq \epsilon$  for all  $k$  sufficiently large then  $\{x_k\}$  is bounded and any limit point  $y^*$  of  $\{x_k\}$  satisfies  $\|y^* - x^*\| \leq \epsilon$ . Hence  $y^* = x^*$ , and thus  $\{x_k\}$  converges to  $x^*$ . This contradiction shows that there is an infinite sequence of indices with  $\|x_k - x^*\| > \epsilon$ . Thus there is a sequence  $\{l_i\}$  such that

$$\|x_{l_i} - x^*\| \leq \epsilon, \quad \|x_{l_{i+1}} - x^*\| > \epsilon.$$

The sequence  $\{x_{l_i}\}$  is bounded. Moreover, if  $y^*$  is a limit point of  $\{x_{l_i}\}$  then  $\|y^* - x^*\| \leq \epsilon$ . Hence  $y^* = x^*$ , and thus the sequence  $\{x_{l_i}\}$  converges to  $x^*$ . In particular,  $\|x_{l_i} - x^*\| \leq \frac{1}{2}\epsilon$  for  $i$  large enough, and therefore

$$\|x_{l_{i+1}} - x_{l_i}\| \geq \|x_{l_{i+1}} - x^*\| - \|x_{l_i} - x^*\| \geq \frac{1}{2}\epsilon.$$

This completes the proof.  $\square$

Up to this point our analysis applies to a fairly general model  $\psi_k$ , but our convergence results assume that the model  $\psi_k$  is the quadratic

$$\psi_k(w) = \langle \nabla f(x_k), w \rangle + \frac{1}{2} \langle w, B_k w \rangle. \tag{6.3}$$

We assume that the sequence  $\{B_k\}$  is bounded; for a quadratic model this is equivalent to assuming that the sequence  $\{\beta_k\}$  is bounded. We also assume that if  $(x^*, B^*)$  is a limit point of the sequence  $\{(x_k, B_k)\}$  then

$$\langle \nabla f(x^*), w \rangle = 0, \quad w \in T(x^*), \quad w \neq 0 \implies \langle w, B^* w \rangle > 0. \tag{6.4}$$

These assumptions on the model  $\psi_k$  are satisfied, for example, if  $B_k = \nabla^2 f(x_k)$  and the limit point  $x^*$  satisfies (6.1). They are also satisfied if  $\{B_k\}$  is a sequence of positive definite matrices with eigenvalues in some fixed interval which does not contain the origin.

We also need an assumption on the step  $s_k$ . This assumption is motivated by the result that  $\langle \nabla f(x_k), s_k(\hat{\alpha}_k) \rangle$  is negative. It is thus reasonable to assume that in addition to the conditions of Section 3, the step  $s_k$  is chosen so that

$$\langle \nabla f(x_k), s_k \rangle \leq 0. \tag{6.5}$$

Any step  $s_k$  that satisfies this restriction is usually called a feasible descent direction.

**Theorem 6.5.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on  $\Omega$ . Assume that the model  $\psi_k$  is the quadratic (6.3), that the sequence  $\{B_k\}$  is bounded, that any limit point of the sequence  $\{(x_k, B_k)\}$  satisfies (6.4), and that the step  $s_k$  satisfies (6.5). If  $x^*$  is an isolated limit point of  $\{x_k\}$  then  $\{x_k\}$  converges to  $x^*$ .*

**Proof.** If  $\{x_k\}$  does not converge to  $x^*$  then Lemma 6.4 shows that there is an infinite sequence  $\mathcal{H}$  such that  $\{x_k\}$  converges to  $x^*$  for  $k \in \mathcal{H}$  and an  $\epsilon > 0$  such that  $\|s_k\| \geq \epsilon$  for  $k \in \mathcal{H}$ . We now prove that if we define a sequence  $\{w_k\}$  by setting

$$w_k = s_k / \|s_k\|, \quad k \in \mathcal{H}, \tag{6.6}$$

then any limit point  $w$  belongs to  $T(x^*)$  and  $\langle \nabla f(x^*), w \rangle = 0$ . Note that  $\|s_k\| \geq \epsilon$  implies that  $x_k + \tau w_k$  belongs to  $\Omega$  for  $\tau$  in  $[0, \epsilon]$ , and hence  $x^* + \tau w$  also belongs to  $\Omega$ . This shows that  $w$  is a feasible direction at  $x^*$  and thus  $w \in T(x^*)$ . Since the first order conditions guarantee that  $-\nabla f(x^*) \in N(x^*)$ , we obtain that  $\langle \nabla f(x^*), w \rangle \geq 0$ . On the other hand, (6.5) implies that  $\langle \nabla f(x^*), w \rangle \leq 0$ .

We have shown that  $w \in T(x^*)$  and that  $\langle \nabla f(x^*), w \rangle = 0$ . Hence, our assumptions on  $\{B_k\}$  imply that there is a limit point  $B^*$  of  $\{B_k\}$  with  $k \in \mathcal{K}$  such that  $\langle w, B^* w \rangle$  is positive. However,  $\psi_k(s_k) \leq 0$  implies that

$$\frac{1}{2} \|s_k\| \langle w_k, B_k w_k \rangle \leq -\langle \nabla f(x_k), w_k \rangle,$$

and since  $w$  is a limit point of  $\{w_k\}$ , we obtain that  $\epsilon \langle w, B^* w \rangle \leq 0$ . This contradiction establishes the result.  $\square$

Assumption (6.5) can be relaxed to allow positive values of  $\langle \nabla f(x_k), s_k \rangle$ . For example, Theorem 6.5 also holds if we assume that

$$\langle \nabla f(x_k), s_k \rangle \leq \nu |\langle \nabla f(x_k), s_k(\alpha_k) \rangle|$$

for some constant  $\nu > 0$ . This claim can be established by noting that assumption (6.5) is only used to guarantee that if  $w$  is a limit point of the sequence  $\{w_k\}$  defined by (6.6), then  $\langle \nabla f(x^*), w \rangle \leq 0$ .

Theorem 6.5 is of interest because  $\Omega$  is allowed to be a general convex set. In the remainder of this section we assume a polyhedral  $\Omega$  and replace (6.5) by the assumption that the step  $s_k$  is chosen so that

$$A(x_k^C) \subset A(x_k + s_k). \tag{6.7}$$

Motivation for this restriction is based on the identification properties of the gradient projection method which suggest that the Cauchy point  $x_k^C$  is a predictor of the optimal active constraints in a neighborhood of a nondegenerate stationary point.

**Theorem 6.6.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on a polyhedral  $\Omega$ . Assume that the model  $\psi_k$  is the quadratic (6.3), that the sequence  $\{B_k\}$  is bounded, that any limit point of the sequence  $\{(x_k, B_k)\}$  satisfies (6.4), and that the step  $s_k$  satisfies (6.7). If  $x^*$  is an isolated limit point of  $\{x_k\}$  and  $x^*$  is nondegenerate then  $\{x_k\}$  converges to  $x^*$ .*

**Proof.** The proof is similar to that of Theorem 6.5. We only need to show that if there is an infinite sequence  $\mathcal{K}$  such that  $\{x_k\}$  converges to  $x^*$  for  $k \in \mathcal{K}$  and an  $\epsilon > 0$  such that  $\|s_k\| \geq \epsilon$  for  $k \in \mathcal{K}$ , then any limit point  $w$  of the sequence  $\{w_k\}$  defined by (6.6) satisfies  $\langle \nabla f(x^*), w \rangle \leq 0$ .

We prove that  $\langle \nabla f(x^*), w \rangle \leq 0$  by establishing that  $-w \in T(x^*)$ . Note that the iterations in  $\mathcal{K}$  are successful, and thus the bound (4.2) implies that

$$f(x_k) - f(x_{k+1}) \geq \mu_4 \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \min \left\{ \Delta_k, \frac{1}{\beta_k} \left[ \frac{\|s_k(\hat{\alpha}_k)\|}{\hat{\alpha}_k} \right] \right\}$$

for all  $k \in \mathcal{K}$ . Since  $\mu_2 \Delta_k \geq \|s_k\| \geq \epsilon$ , and since  $\{\beta_k\}$  is bounded, this shows that

$$\lim_{k \in \mathcal{K}} \frac{\|x_k^C - x_k\|}{\hat{\alpha}_k} = 0,$$

and hence Lemma 5.1 implies that

$$\lim_{k \in \mathcal{K}} \|\nabla_{\Omega} f(x_k^C)\| = 0.$$

Since  $\{x_k^C\}$  also converges to  $x^*$ , Theorem 2.2 implies that

$$A(x^*) = A(x_k^C) \subset A(x_k + s_k), \quad k \in \mathcal{K}.$$

This implies that  $-s_k \in T(x^*)$ , and hence  $-w \in T(x^*)$  as desired.  $\square$

The assumptions of Theorem 6.6 can be satisfied in a number of ways. The following result illustrates the relationship between the assumptions in Theorem 6.6 and will be useful in Section 7.

**Corollary 6.7.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be twice continuously differentiable on a polyhedral  $\Omega$ , and let  $\nabla^2 f$  be bounded on the level set*

$$\mathcal{L}(x_0) = \{x \in \Omega: f(x) \leq f(x_0)\}. \tag{6.8}$$

*Assume that the model  $\psi_k$  is the quadratic*

$$\psi_k(w) = \langle \nabla f(x_k), w \rangle + \frac{1}{2} \langle w, \nabla^2 f(x_k) w \rangle,$$

*and that the step  $s_k$  satisfies (6.7). If  $\{x_k\}$  has a limit point  $x^*$  which is nondegenerate and satisfies (6.2), then  $\{x_k\}$  converges to  $x^*$ .*

**Proof.** Theorem 5.4 guarantees that any limit point of  $\{x_k\}$  is a stationary point of problem (1.2), and thus Theorems 6.1 and 6.3 show that  $x^*$  is an isolated limit point of  $\{x_k\}$ . The result now follows from Theorem 6.6.  $\square$

Note that Theorem 6.5 shows that we can drop the assumption that  $x^*$  is nondegenerate and establish a similar result for a general convex set  $\Omega$  if we assume that the step  $s_k$  satisfies (6.5) instead of (6.7).

## 7. Identification of constraints and rates of convergence

We have shown that the trust region method of Section 3 converges under reasonable conditions. The main result of this section shows that if the trust region iterates  $\{x_k\}$  converge to a nondegenerate stationary point  $x^*$ , then the active set at  $x^*$  is identified in a finite number of iterations.

We begin the development with a technical result which will be used to show that if an iterate is close to  $x^*$  and in the same face as  $x^*$ , then the Cauchy step remains in this face. Recall that a face  $\Omega_F$  such that  $x^* \in \text{ri}(\Omega_F)$  is defined by (2.4) with  $\mathcal{A} = \mathcal{A}(x^*)$ , and that the relative interior of this face has the explicit representation (2.5).

**Lemma 7.1.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on a polyhedral  $\Omega$ , and assume that  $x^*$  is a nondegenerate stationary point. If  $\Omega_F$  is the face of  $\Omega$  such that  $x^* \in \text{ri}(\Omega_F)$  then for any constant  $\gamma_3 > 0$  there is a neighborhood  $S(x^*)$  such that for  $\alpha \in (0, \gamma_3]$ ,*

$$x \in \text{ri}(\Omega_F) \cap S(x^*) \Rightarrow P(x - \alpha \nabla f(x)) \in \text{ri}(\Omega_F).$$

**Proof.** Theorem 2.3 shows that

$$x^* - \gamma_3 \nabla f(x^*) \in \text{ri}(\Omega_F) + \text{ri}(N(\Omega_F)) = \text{int}\{\Omega_F + N(\Omega_F)\},$$

and thus we can choose  $S(x^*)$  such that for  $x \in S(x^*)$ ,

$$x - \gamma_3 \nabla f(x) \in \text{int}\{\Omega_F + \text{ri}(N(\Omega_F))\} = \text{ri}\{\Omega_F + N(\Omega_F)\}. \tag{7.1}$$

A standard result in convex analysis (see, for example, Rockafellar, 1970, Theorem 6.1) guarantees that the relative interior of a line segment belongs to the relative interior of a convex set  $\Gamma$  if one endpoint belongs to  $\Gamma$  and the other endpoint belongs to the relative interior of  $\Gamma$ . Hence, (7.1) implies that for  $\alpha \in (0, \gamma_3]$ ,

$$\left(1 - \frac{\alpha}{\gamma_3}\right)x + \frac{\alpha}{\gamma_3}(x - \gamma_3 \nabla f(x)) = x - \alpha \nabla f(x) \in \text{ri}\{\Omega_F + N(\Omega_F)\}. \tag{7.2}$$

Note that (2.7) shows that  $P(x + v) = x$  if  $v \in N(x)$  and thus

$$P[\text{ri}\{\Omega_F + N(\Omega_F)\}] = P[\text{ri}(\Omega_F) + \text{ri}(N(\Omega_F))] \subset \text{ri}(\Omega_F).$$

In view of (7.2), this completes the proof.  $\square$

The identification properties require an additional assumption on the step because if we only require (3.8) then  $(1 - \epsilon)s_k$  also satisfies (3.8) for  $\epsilon > 0$  sufficiently small. We now show that an appropriate assumption is that the step  $s_k$  satisfies (6.7).

**Theorem 7.2.** *Assume that  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable on a polyhedral  $\Omega$ , that  $\{\beta_k\}$  is bounded, that (5.3) holds whenever (5.2) is convergent, and that the step  $s_k$  satisfies (6.7). If  $\{x_k\}$  converges to a nondegenerate point  $x^*$  then*

$$\lim_{k \rightarrow \infty} \|\nabla_{\Omega} f(x_k)\| = 0.$$

Moreover, there is an index  $k_0 > 0$  such that

$$\mathcal{A}(x_k) = \mathcal{A}(x^*), \quad s_k \in N(x^*)^\perp, \quad k \geq k_0. \tag{7.3}$$

**Proof.** First note that the definition of the active set  $\mathcal{A}(x)$  shows that  $\mathcal{A}(x) \subset \mathcal{A}(x^*)$  whenever  $x \in \Omega$  is sufficiently close to  $x^*$ . Moreover, if  $\Omega_F$  is the face of  $\Omega$  such that  $x^* \in \text{ri}(\Omega_F)$  then  $\mathcal{A}(x) = \mathcal{A}(x^*)$  if and only if  $x \in \text{ri}(\Omega_F)$ .

Theorem 5.5 shows that there is a sequence  $\mathcal{H}$  of successful iterates such that  $\{\nabla_{\Omega} f(x_k^C)\}$  converges to zero for  $k \in \mathcal{H}$ , and thus Theorem 2.2 guarantees that  $\mathcal{A}(x_k^C) = \mathcal{A}(x^*)$  for all  $k \in \mathcal{H}$  sufficiently large. Now note that assumption (6.7) on the step yields that

$$\mathcal{A}(x^*) = \mathcal{A}(x_k^C) \subset \mathcal{A}(x_k + s_k) = \mathcal{A}(x_{k+1}) \subset \mathcal{A}(x^*),$$

and hence,  $x_{k+1} \in \text{ri}(\Omega_F)$  for all  $k \in \mathcal{H}$  sufficiently large. We have shown that for any  $\epsilon > 0$  there is an index  $k_0 > 0$  such that  $x_{k_0} \in \text{ri}(\Omega_F)$  and  $\|x_k - x^*\| \leq \epsilon$  for  $k \geq k_0$ . We now prove that  $x_k \in \text{ri}(\Omega_F)$  for all  $k \geq k_0$ .

Lemma 7.1 guarantees that  $\epsilon$  can be chosen so that if  $x_k \in \text{ri}(\Omega_F)$  then  $x_k^C \in \text{ri}(\Omega_F)$ . Note that  $\mathcal{A}(x_k^C) = \mathcal{A}(x^*)$  for any such index  $k$ . We claim that  $x_{k+1} \in \text{ri}(\Omega_F)$ . There is nothing to prove if  $x_{k+1} = x_k$ . Otherwise assumption (6.7) on the step implies that

$$\mathcal{A}(x^*) = \mathcal{A}(x_k^C) \subset \mathcal{A}(x_{k+1}) \subset \mathcal{A}(x^*).$$

Hence,  $x_{k+1} \in \text{ri}(\Omega_F)$  as claimed. This shows that  $x_k \in \text{ri}(\Omega_F)$  for all  $k \geq k_0$ , and thus  $\mathcal{A}(x_k) = \mathcal{A}(x^*)$ . Moreover, Lemma 7.1 implies that  $\mathcal{A}(x_k^C) = \mathcal{A}(x^*)$ . Hence

$$\mathcal{A}(x_k) = \mathcal{A}(x^*) = \mathcal{A}(x_k^C) \subset \mathcal{A}(x_k + s_k),$$

and this yields that  $s_k \in N(x^*)^\perp$  for  $k \geq k_0$ . Finally,  $\{\nabla_{\Omega} f(x_k)\}$  converges to zero because  $x_k$  and  $x^*$  lie in  $\text{ri}(\Omega_F)$  and  $\nabla_{\Omega} f(\cdot)$  is continuous on  $\text{ri}(\Omega_F)$ .  $\square$

Theorem 7.2 shows that eventually all the iterates lie in the relative interior of the face  $\Omega_F$  such that  $x^* \in \text{ri}(\Omega_F)$ . Moreover, since  $s_k \in N(x^*)^\perp$ , all the trial steps  $x_k + s_k$  belong to  $\Omega_F$ . This result suggests that we can now apply results from unconstrained minimization and obtain, for example, rate of convergence results. However, our choice of  $\alpha_k$  is more general than the usual choice in unconstrained minimization, and this prevents the immediate application of the unconstrained minimization results. This difficulty is resolved by proving that the bound (4.1) on the predicted decrease can be expressed in terms of the projection of the gradient.

**Lemma 7.3.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable on a polyhedral  $\Omega$ , and assume that  $x^*$  is a nondegenerate stationary point. If  $\Omega_F$  is the face of  $\Omega$  such that  $x^* \in \text{ri}(\Omega_F)$  and  $P_*$  is the projection into  $N(x^*)^\perp$ , then for any constant  $\gamma_3 > 0$  there is a neighborhood  $S(x^*)$  such that for  $\alpha \in (0, \gamma_3]$  and  $x \in \text{ri}(\Omega_F) \cap S(x^*)$ ,*

$$\frac{P(x - \alpha \nabla f(x)) - x}{\alpha} = -P_*[\nabla f(x)] = \nabla_{\Omega} f(x).$$

**Proof.** If  $S(x^*)$  is as in Lemma 7.1 and  $x \in \text{ri}(\Omega_F) \cap S(x^*)$ , then

$$N[P(x - \alpha \nabla f(x))] \subset N(\Omega_F)$$

for  $\alpha \in (0, \gamma_3]$ . Since the characterization (2.6) of the projection implies that  $z - P(z)$  belongs to  $N[P(z)]$  for any  $z \in \mathbb{R}^n$ , we obtain that

$$(x - \alpha \nabla f(x)) - P(x - \alpha \nabla f(x)) \in N(\Omega_F).$$

This expression can be written in the form

$$-\nabla f(x) = \frac{P(x - \alpha \nabla f(x)) - x}{\alpha} + v,$$

where  $v \in N(x^*)$ . Moreover, Lemma 7.1 implies that  $P(x - \alpha \nabla f(x))$  belongs to  $\text{ri}(\Omega_F)$  and therefore,  $P(x - \alpha \nabla f(x)) - x$  lies in  $N(x^*)^\perp$ . Since  $P_*(v) = 0$  we obtain

$$-P_*[\nabla f(x)] = \frac{P(x - \alpha \nabla f(x)) - x}{\alpha}.$$

This relationship holds for  $\alpha \in (0, \gamma_3]$ , and thus an application of Lemma 2.5 completes the proof.  $\square$

An immediate consequence of Theorem 7.2 and Lemma 7.3 is that if the sequence  $\{x_k\}$  converges to a nondegenerate point  $x^*$  then

$$\|P_*[\nabla f(x_k)]\| = \|s_k(\hat{\alpha}_k)\|/\hat{\alpha}_k,$$

and thus the bound (4.1) on the predicted decrease implies that

$$-\psi_k(s_k) \geq \mu_0^2 \mu_3 \|P_*[\nabla f(x_k)]\| \min \left\{ \Delta_k, \frac{1}{\beta_k} \|P_*[\nabla f(x_k)]\| \right\}.$$

This is the appropriate bound on the predicted decrease when the iterates lie in  $\text{ri}(\Omega_F)$ . Thus, we can now apply unconstrained minimization results (see, for example, Moré, 1983; Schultz, Schnabel and Byrd, 1985) to the trust region method of Section 3. The following result is obtained by combining the above observations with Corollary 6.7, Theorem 7.2 and Theorem 4.19 of Moré (1983).

**Theorem 7.4.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be twice continuously differentiable on a polyhedral  $\Omega$ , and let  $\nabla^2 f$  be bounded on the level set (6.8). Assume that the model  $\psi_k$  is the quadratic*

$$\psi_k(w) = \langle \nabla f(x_k), w \rangle + \frac{1}{2} \langle w, \nabla^2 f(x_k) w \rangle, \tag{7.4}$$

*and that the step  $s_k$  satisfies (6.7). If  $\{x_k\}$  has a limit point  $x^*$  which is nondegenerate and satisfies (6.2), then  $\{x_k\}$  converges to  $x^*$  and (7.3) holds for some index  $k_0 > 0$ . Moreover, all iterations are eventually successful and  $\{\Delta_k\}$  is bounded away from zero.  $\square$*

Rate of convergence results can be obtained under the assumptions of Theorem 7.4 by applying standard unconstrained results. We motivate our choice of step by first noting that if  $\psi_k$  is the quadratic (7.4) and  $P_k$  is the orthogonal projection into  $N(x_k)^\perp$ , then

$$\psi_k(P_k w) = \langle g_k, w \rangle + \frac{1}{2} \langle w, B_k w \rangle,$$

where

$$B_k = P_k \nabla^2 f(x_k) P_k, \quad g_k = P_k \nabla f(x_k).$$

A truncated Newton method restricted to  $N(x_k)^\perp$  satisfies

$$\|g_k + B_k s_k\| \leq \xi_k \|g_k\|, \quad \xi_k \in (0, 1). \tag{7.5}$$

A truncated Newton method can be obtained by either iterative or direct methods. See, for example, the discussion in Moré (1983). There is no guarantee that if  $s_k$  satisfies (7.5) then  $x_k + s_k$  belongs to the feasible set  $\Omega$ , and thus we only require that (7.5) hold on selected iterations. However, note that under the assumptions of Theorem 7.4 any step which satisfies (7.5) converges to zero, and since  $\mathcal{A}(x_k) = \mathcal{A}(x^*)$ , the trial step  $x_k + s_k$  is eventually feasible.

**Theorem 7.5.** *Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be twice continuously differentiable on a polyhedral  $\Omega$ , and let  $\nabla^2 f$  be bounded on the level set (6.8). Assume that the model  $\psi_k$  is the quadratic (7.4) and that the step  $s_k$  satisfies (6.7). Moreover, assume that there is a constant  $\mu_* > 0$  such that if  $\|s_k\| \leq \mu_* \Delta_k$  and  $\mathcal{A}(x_k) = \mathcal{A}(x_k + s_k)$ , then  $s_k$  satisfies (7.5). If  $\{x_k\}$  has a limit point  $x^*$  which is nondegenerate and satisfies (6.2), then  $\{x_k\}$  converges Q-linearly to  $x^*$  provided  $\xi^* < 1$  where*

$$\xi^* = \limsup_{k \rightarrow \infty} \xi_k.$$

If  $\xi^* = 0$  then  $\{x_k\}$  converges Q-superlinearly to  $x^*$ .

**Proof.** Theorem 7.4 guarantees that  $\{\Delta_k\}$  is bounded away from zero and that  $\{s_k\}$  converges to zero. Hence, we can assume that if  $k_0$  is as in Theorem 7.4, then  $\|s_k\| \leq \mu_* \Delta_k$  for  $k \geq k_0$ . In particular,  $s_k$  satisfies (7.5) for  $k \geq k_0$ . The result is now fairly easy to obtain. If  $P_*$  is the projection into  $N(x^*)^\perp$  then  $P_k = P_*$  for  $k \geq k_0$  and thus (7.5) implies that

$$\limsup_{k \rightarrow \infty} \frac{\|P_* \nabla f(x_{k+1})\|}{\|P_* \nabla f(x_k)\|} \leq \limsup_{k \rightarrow \infty} \xi_k = \xi^*.$$

Since  $P_* \nabla^2 f(x^*) P_*$  is positive definite on  $N(x^*)^\perp$ , this proves that  $\{x_k\}$  converges Q-linearly if  $\xi^* < 1$  and that  $\{x_k\}$  converges Q-superlinearly if  $\xi^* = 0$ .  $\square$

If  $\nabla^2 f$  is Lipschitz continuous near  $x^*$  and if we choose  $\xi_k = 0$  in (7.5), then  $\{x_k\}$  converges Q-quadratically to  $x^*$  because the iteration eventually reduces to Newton's method on  $N(x^*)^\perp$ . The Newton step  $s_k$  can be obtained by determining a basis  $Z_k$  for  $N(x_k)^\perp$ , solving the null-space equations

$$(Z_k^T \nabla^2 f(x_k) Z_k) s_k^N = -Z_k^T \nabla f(x_k)$$

for  $s_k^N$  and setting  $s_k = Z_k s_k^N$ . Note that although  $s_k^N$  is dependent on the choice of  $Z_k$ , the step  $s_k$  is independent of the choice of basis.

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