JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 159, 485–499 (1991)

Weak Directional Closedness and Generalized Subdifferentials

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Submitted by Frank H. Clark

Received August 3, 1989

Spingarn introduced the notion of a submonotone operator and showed that the Clarke subdifferential is submonotone if and only if it is semismooth (in the sense of Mifflin) and regular (in the sense of Clarke). In this article a property of operators referred to as weak directional closedness (WDC) is introduced. The WDC property is used to extend Spingarn's result to a broad class of generalized subdifferentials for locally Lipschitz functions. Two members of this class of subdifferentials are the Clarke subdifferential, which is always WDC, and the Michel–Penot subdifferential, which may or may not be WDC. A subdifferential that is WDC and is contained in the Clarke subdifferential is constructed. It is shown that this subdifferential coincides with the Michel–Penot subdifferential whenever the Michel–Penot subdifferential is WDC and submonotone.

1. INTRODUCTION

Over the past 30 years the importance of nonsmooth functions for modern optimization theory has been well established. Consequently, the variational properties of these functions have been extensively studied. Perhaps the most powerful tool in this emerging method of analysis is the

* This author's work is supported in part by the National Science Foundation Grant DMS-8803206.

† This author's work is supported in part by the Australian Research Council.

0022-247X/91 $3.00

concept of a subdifferential. Several types of subdifferentials have been proposed. Most of these extend the now classical notion of a subdifferential from convex analysis [19]. The first and most successful of these new subdifferentials is due to Clarke [6]. Unfortunately, the Clarke subdifferential is not entirely satisfactory in certain applications and so many others have been proposed [8, 10–12, 15, 16, 22, 23]. Recently there has been an effort to unify the theory of subdifferentials and to classify them according to their properties and their utility for various applications. Some excellent references along these lines are Ioffe [10], Ward [23], and Treiman [21, 22]. The present article is written in the spirit of this work. We consider Spingarn's [20] notion of a submonotone operator and examine its role in the context of operators possessing properties shared by most generalized subdifferentials. In particular, we extend the following result due to Spingarn [20].

THEOREM 1 (Spingarn). *A locally Lipschitz function $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ is both regular and semismooth if and only if its Clarke subdifferential is submonotone.*

The terms *regular*, *semismooth*, and *submonotone* are defined in the next section wherein all of the necessary basic ingredients are reviewed. As an application of the concepts introduced in this section, we provide a characterization of Pshenichnyi's notion of *quasidifferentiability* [16]. In Section 3 we introduce the notion of a weak directionally closed (WDC) operator and establish our extension of Spingarn's result. In Section 4, a subdifferential that is always WDC is constructed. It is shown that this subdifferential is contained in the Clarke subdifferential and coincides with the Michel–Penot subdifferential whenever the Michel–Penot subdifferential is WDC and submonotone.

2. THE BASICS

A multifunction $T\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be

(a) *compact-valued* if $T(x)$ is compact for every $x \in \operatorname{dom}(T)$, where

$$\operatorname{dom}(T) := \{x \in \mathbb{R}^n : T(x) \neq \varnothing\},$$

(b) *convex-valued* if $T(x)$ is convex for every $x \in \operatorname{dom}(T)$,

(c) *locally bounded* if for each $\bar{x} \in \operatorname{dom}(T)$ there is a neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that the set

$$\bigcup_{x \in U_{\bar{x}}} T(x)$$

is bounded, and

(d) *closed* if $\operatorname{graph}(T) := \{(x, y) : x \in \operatorname{dom}(T), y \in T(x)\}$ is a closed set.

We denote by $\mathscr{L}$ the set of all locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$ and by $\mathscr{M}$ the set of all compact-valued, locally bounded multifunctions mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. In this section we consider mappings ∂ from $\mathscr{L}$ to $\mathscr{M}$ known as *generalized subdifferentials*. Two such generalized subdifferentials are the Clarke subdifferential [6] and the Michel–Penot subdifferential [12]. We denote the Clarke and Michel–Penot subdifferential operators by ∂° and $\partial^{\diamond}$, respectively. For a given function $f \in \mathscr{L}$ we recall that the Clarke and Michel–Penot generalized subdifferentials are given by the expressions

$$\partial^{\circ} f(x) := \{u \in \mathbb{R}^n : \langle u, h\rangle \leqslant \limsup_{y \to x, t \downarrow 0} [f(y + th) - f(y)]/t, \forall h \in \mathbb{R}^n\}$$

and

$$\partial^{\diamond} f(x) := \{u : \langle u, h\rangle \leqslant \sup_{k \in \mathbb{R}^n} \{\limsup_{t \downarrow 0} [f(x + th + tk) - f(x + tk)]/t\}, \forall h \in \mathbb{R}^n\},$$

respectively. It is well-known that for each $f \in \mathscr{L}$ one has $\operatorname{dom}(\partial^{\circ} f) = \operatorname{dom}(\partial^{\diamond} f) = \mathbb{R}^n$ and

$$\partial^{\diamond} f(x) \subseteq \partial^{\circ} f(x) \tag{1}$$

for all $x \in \mathbb{R}^n$. Moreover, both the Clarke and the Michel–Penot subdifferentials are convex-valued. Examples of other subdifferentials are the *proximal subdifferential* [18] denoted ∂_2 and the *Dini subdifferential* [11, 15] denoted ∂^-. For $f \in \mathscr{L}$ one has $u \in \partial_2 f(\bar{x})$ if and only if

$$\liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle u, x - \bar{x}\rangle}{|x - \bar{x}|^2} > -\infty,$$

and $u \in \partial^- f(\bar{x})$ if and only if

$$\liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle u, x - \bar{x}\rangle}{|x - \bar{x}|} \geqslant 0.$$

We note that $\partial^- f(x)$ has the alternative description

$$\{u \in \mathbb{R}^n : \langle u, h\rangle \leqslant f^-(x; h), \forall h \in \mathbb{R}^n\},$$

where $f^-(x; h)$ is the lower Dini directional derivative,

$$f^-(x; h) := \liminf_{\substack{t \downarrow 0 \\ v \to h}} \frac{f(x + tv) - f(x)}{t}.$$

Although it is not necessarily the case that either of the sets $\operatorname{dom}(\partial_2 f)$ or $\operatorname{dom}(\partial^- f)$ are all of $\mathbb{R}^n$, it is the case that these sets are dense in $\mathbb{R}^n$. From ∂_2 and ∂^- one can construct the *limiting proximal* and *limiting Dini* subdifferentials as

$$\hat{\partial}_2 f(x) := \{u \in \mathbb{R}^n : u = \lim_i u_i,\ u_i \in \partial_2 f(x_i),\ \partial_2 f(x_i) \neq \emptyset,\ x_i \to x\}$$

and

$$\hat{\partial}^- f(x) := \{u \in \mathbb{R}^n : u = \lim_i u_i,\ u_i \in \partial^- f(x_i),\ \partial^- f(x_i) \neq \emptyset,\ x_i \to x\},$$

respectively. The construction of other generalized subdifferentials is discussed in [10, 16, 8, 22, 23].

We begin our discussion with a few elementary observations. First note that $\mathcal{L}$ is a linear space under the usual notions of addition and scalar multiplication of functions, and $\mathcal{M}$ is a monoid over the reals under the following definitions: Let $U, V \in \mathcal{M}$ and let $\lambda > 0$, then

(i) $U + V \in \mathcal{M}$ is defined pointwise by

$$(U+V)(x) := \{u + v : u \in U(x),\ v \in V(x)\},$$

for all $x \in \mathbb{R}^n$, and

(ii) $\lambda U \in \mathcal{M}$ is also defined pointwise by

$$\lambda U(x) := \{\lambda u : u \in U(x)\},$$

for all $x \in \mathbb{R}^n$.

In (i) above, we employ the convention that $S + \emptyset = \emptyset$ for any subset S of $\mathbb{R}^n$. Denote by $\tilde{\mathcal{M}}$ the submonoid of $\mathcal{M}$ consisting of all convex-, compact-valued, multifunctions mapping $\mathbb{R}^n$ to $\mathbb{R}^n$. For every $T \in \mathcal{M}$ and $x \in \mathbb{R}^n$ we define $\overline{\text{co}}\, T(x)$ to be the closed convex hull of $T(x)$. The operator $\overline{\text{co}}: \mathcal{M} \to \tilde{\mathcal{M}}$ so defined is a projection onto $\tilde{\mathcal{M}}$. Indeed, $\overline{\text{co}}$ is linear and idempotent. Furthermore, for each $T \in \mathcal{M}$ the projection of T onto $\tilde{\mathcal{M}}$, $\overline{\text{co}}\, T$, is completely determined by its associated support functional

$$\psi^*(\cdot \mid T(x)) := \sup\{\langle z, \cdot \rangle : z \in T(x)\},$$

where $\psi^*(\cdot \mid T(x))$ maps from $\mathbb{R}^n$ to $\mathbb{R}^n \cup \{\pm\infty\}$ taking the value $-\infty$ whenever $T(x) = \emptyset$. Given a mapping $\partial: \mathcal{L} \mapsto \mathcal{M}$ we define $\overline{\text{co}}\, \partial: \mathcal{L} \to \tilde{\mathcal{M}}$ by $\overline{\text{co}}\, \partial(f) := \overline{\text{co}}(\partial f)$ for each $f \in \mathcal{L}$. It is well known that

$$\overline{\text{co}}\, \hat{\partial}_2 f = \overline{\text{co}}\, \hat{\partial}^- f = \partial^\circ f. \tag{2}$$

In order for a mapping $\partial: \mathscr{L} \mapsto \mathscr{M}$ to be considered a generalized subdifferential it should possess certain properties that generalize standard properties of the differential calculus. A few such properties are:

positive homogeneity: $\partial(\lambda f) = \lambda\, \partial f$ for all $\lambda > 0$ and $f \in \mathscr{L}$.

symmetry: $\partial(-f) = -\partial f$ for all $f \in \mathscr{L}$.

weak additivity: $\partial(f + \langle z, \cdot \rangle) = \partial f + z$ for all $f \in \mathscr{L}$ and $z \in \mathbb{R}^n$.

lower extremality: $f^-(x; h) \leqslant \psi^*(h \mid \partial f(x))$ for all $x, h \in \mathbb{R}^n$ and all $f \in \mathscr{L}$.

The properties listed above do not yield as rich a calculus as one might desire for a generalized subdifferential. For example they do not yield the restricted calculus discussed in Ioffe [10]. Nontheless, they give the flavor of the type of conditions that are usually satisfied.

THEOREM 2. *Let $\partial: \mathscr{L} \mapsto \mathscr{M}$, $f \in \mathscr{L}$, and $x \in \mathbb{R}^n$.*

(a) Minimality Conditions: *If ∂ is lower extremal and f attains a local minimum at x, then $0 \in \overline{\text{co}}\, \partial f(x)$.*

(b) Optimality Conditions: *If ∂ is lower extremal and symmetric, and f attains either a local minimum or local maximum at x, then $0 \in \overline{\text{co}}\, \partial f(x)$.*

(c) The Mean Value Theorem: *Let us further suppose that $\text{dom}(\partial f) = \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is such that $y \neq x$. If ∂ is symmetric, weakly additive, and lower extremal then there exists $z := \lambda x + (1 - \lambda) y$ with $0 < \lambda < 1$ and $u \in \overline{\text{co}}\, \partial f(z)$ such that*

$$f(y) - f(x) = \langle u, y - x \rangle.$$

Remark. Observe that if x is a local minimum for f, then it is necessarily the case that $0 \in \partial^- f(x)$. Hence if ∂ is lower extremal, then $\partial f(x) \neq \varnothing$ and $0 \in \overline{\text{co}}\, \partial f(x)$.

We do not pause to prove this result since the proof follows standard arguments found in the literature. Nonetheless, we note that the Lebourg Mean Value Theorem for the Clarke subdifferential and the Borwein–Fitzpatrick–Giles Mean Value Theorem [3] for the Michel–Penot subdifferential are special versions of the Mean Value Theorem given above. We should also mention that any subdifferential that does not satisfy the conditions given above is probably not particularly useful. Both the Clarke and Michel–Penot subdifferentials satisfy these conditions.

A sequence $\{x_j\}$ is said to converge to $x \in \mathbb{R}^n$ in the direction $h \in \mathbb{R}^n$, written $x_j \xrightarrow[h]{} x$ if $h \neq 0$, $x_j \neq x$, and $(x_j - x)/|x_j - x| \to h/|h|$. Given $\partial: \mathscr{L} \mapsto \mathscr{M}$ and $f \in \mathscr{L}$, we say that f is ∂-semismooth at $x \in \mathbb{R}^n$ if $x_j \xrightarrow[h]{} x$, $\{x_j\} \subset \text{dom}(\partial f)$, and $v_j \in \partial f(x_j)$ imply that $\langle v_j, h \rangle \to f'(x; h)$ for all $h \in \mathbb{R}^n$.

We say that $f \in \mathscr{L}$ is ∂-semismooth if f is ∂-semismooth at each point in $\mathbb{R}^n$. From (1), we see that if f is ∂°-semismooth at x, then f is $\partial^\diamond$-semismooth at x. The notion of semismoothness was first introduced by Mifflin in [13, 14] for the Clarke subdifferential. Semismoothness is a rather useful property. For example, if $f \in \mathscr{L}$ is ∂°-semismooth, then $\partial^\circ f(x)$ is single-valued almost everywhere [17]. Also see [4, 5].

It turns out that ∂-semismoothness is the same as ∂°-semismoothness under mild assumptions. This result is easily established with the aid of Rademacher's Theorem, Caratheodory's Theorem, and the fact that

$$\partial^\circ f(x) = \overline{\mathrm{co}}\{u \in \mathbb{R}^n : u = \lim_{i \to \infty} \nabla f(x_i), \text{ where } x_i \to x \text{ and } \{x_i\} \subset D\}, \tag{3}$$

where

$$D := \{x \in \mathbb{R}^n : f'(x) \text{ exists}\}.$$

We record this fact in the following proposition.

PROPOSITION 3. *Let $\partial : \mathscr{L} \mapsto \mathscr{M}$. If ∂ is lower extremal and for every $f \in \mathscr{L}$ and $x \in \mathbb{R}^n$, $\partial f(x) \subseteq \partial^\circ f(x)$, then ∂-semismoothness is the same as ∂°-semismoothness.*

Therefore, we simply call ∂°-semismoothness semismoothness as in Mifflin [13, 14]. In particular, $\partial^\diamond$-semismoothness is the same as semismoothness.

Another notion that we make use of is that of ∂-regularity. Given $\partial : \mathscr{L} \mapsto \mathscr{M}$, we say that $f \in \mathscr{L}$ is ∂-regular at $x \in \mathbb{R}^n$ if the usual directional derivative

$$f'(x; h) := \lim_{t \downarrow 0} [f(x + th) - f(x)]/t$$

exists and equals $\psi^*(h \mid \partial f(x))$ for all $h \in \mathbb{R}^n$. Many of the calculus rules hold for the Clarke and Michel–Penot subdifferentials if the corresponding notion of regularity is satisfied [7, 1]. Regularity with respect to the Michel–Penot subdifferential was called *semiregularity* in [1]. A related property is studied by Borwein in [2]. By (1), $\partial^\diamond$-regularity is weaker than ∂°-regularity.

Regularity with respect to the Michel–Penot subdifferential is closely related to the Pshenichnyi's notion of *quasidifferentiability* [16]. Recall that $f \in \mathscr{L}$ is said to be quasidifferentiable at $x \in \mathbb{R}^n$ if there is a closed convex set K in $\mathbb{R}^n$ such that $f'(x; \cdot) = \psi^*(\cdot \mid K)$. It is easy to see that this property is weaker than ∂°-regularity. Spingarn gives in [20] the following example in which f is semismooth and quasidifferentiable and yet f is not ∂°-regular.

EXAMPLE 4 (Spingarn). Let $f: \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by

$$\begin{aligned} f(x, y) &= 0 && \text{if } \ x \leqslant 0; \\ f(x, y) &= x^2/4 && \text{if } \ x > 0, \quad |y| \geqslant x^2/2; \\ f(x, y) &= |y| - y^2/x^2 && \text{if } \ x > 0, \quad |y| < x^2/2. \end{aligned}$$

The function f is locally Lipschitz, semismooth, and quasidifferentiable. But it is not ∂°-regular at the origin.

It truns out that quasidifferentiability is the same as $\partial^{\diamond}$-regularity.

PROPOSITION 5. *A function $f \in \mathscr{L}$ is quasidifferentiable at $x \in \mathbb{R}^n$ if and only if f is $\partial^{\diamond}$-regular at x.*

Proof. If f is $\partial^{\diamond}$-regular at x, then f is quasidifferentiable at x with $K = \partial^{\diamond} f(x)$. Conversely, if f is quasidifferentiable, then

$$f^{\diamond}(x; \cdot) = \sup_{k \in \mathbb{R}^n} [f'(x; k + \cdot) - f'(x; k)] = f'(x; \cdot);$$

i.e., f is $\partial^{\diamond}$-regular at x. ∎

We conclude this section by introducing a variation on Spingarn's notion of a submonotone operator. A multifunction $T: \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be *submonotone* at $x \in \mathbb{R}^n$ if

$$\liminf_{\substack{y \to x,\, y \neq x \\ u \in T(x),\, v \in T(y)}} \langle v - u, y - x \rangle / |y - x| \geqslant 0. \tag{4}$$

In [20], Spingarn requires that T also be locally bounded, convex-valued, and closed. We do not make these restrictions since we wish to study operators that do not necessarily have these properties. We say that the operator T is submonotone if it is submonotone at every $x \in \operatorname{dom}(T)$ it is *maximal* submonotone if there is no other submonotone operator $\hat{T}: \mathbb{R}^n \mapsto \mathbb{R}^n$ such that graph(T) strictly contained in graph$(\hat{T})$. Note that every maximal submonotone operator is necessarily convex-valued; indeed, this follows immediately from Caratheodory's Theorem and the definition.

3. WEAK DIRECTIONAL CLOSEDNESS AND GENERALIZED SUBDIFFERENTIALS

Recall that $T \in \mathscr{M}$ is closed if graph(T) is a closed set. One consequence of closedness is that

$$\limsup_{y \xrightarrow[h]{} x} \psi^*(h \mid T(y)) \leqslant \psi^*(h \mid T(x)), \tag{5}$$

for all x, $h \in \mathbb{R}^n$. One the other hand, an operator $T \in \mathscr{M}$ can satisfy (5) without T being closed. For example, the operator $\partial^{\diamond} f$, where f is the function defined in Example 4, satisfies (5) even though it is not closed. However, in general, the Michel–Penot subdifferential of a locally Lipschitz function does not satisfy (5) as illustrated by the following example.

EXAMPLE 6. Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be defined by

$$f(x) := x^2 \sin(1/x), \qquad \text{for} \quad x \neq 0;$$

and $f(0) = 0$. This function is everywhere differentiable. Consequently, $\partial^{\diamond} f(x) = \{f'(x)\}$ for all $x \in \mathbb{R}$. It is easily verified that (5) does not hold for $T = \partial^{\diamond} f$, $h = 1$, and $x = 0$.

If an operator $T \in \mathscr{M}$ satisfies (5) at a point $x \in \mathbb{R}^n$ for all $h \in \mathbb{R}^n$, then we say that T is weak directionally closed (or WDC) at x. If T is WDC at every point $x \in \operatorname{dom}(T)$, then we simply say that T is WDC. The WDC property can be used to establish the maximality of a submonotone operator.

PROPOSITION 7. *Let $T\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ be a locally bounded multifunction. If T is submonotone, convex-valued, and WDC, then T is maximal submonotone.*

Proof. Suppose that the result is false. Then there exists $x, u \in \mathbb{R}^n$ with u not in $T(x)$ such that for any $h \in \mathbb{R}^n$

$$\begin{aligned} 0 &\leqslant \liminf_{\substack{y \to x,\, y \neq x \\ v \in T(y)}} \left\langle v - u, \frac{y - x}{|y - x|} \right\rangle \\ &\leqslant \limsup_{y \xrightarrow[h]{} x} \psi^*(h \mid T(y)) - \langle u, h \rangle, \end{aligned}$$

where the second inequality follows from the local boundedness of T. Consequently,

$$\langle u, h \rangle \leqslant \limsup_{y \xrightarrow[h]{} x} \psi^*(h \mid T(y)) \leqslant \psi^*(h \mid T(x))$$

for all $h \in \mathbb{R}^n$ since T is WDC. But then $u \in T(x)$ since T is convex-valued. This is the contradiction that establishes the result. ∎

Let $\partial\colon \mathscr{L} \mapsto \mathscr{M}$. We assume that ∂ possesses properties that are associated with the notion of a generalized subdifferential. Specifically, we assume throughout the remainder of this section that ∂ is symmetric, weakly subadditive, and lower extremal. Thus, for each $f \in \mathscr{L}$, ∂f satisfies the properties described in Theorem 2. By employing the WDC property we are able to obtain the following extension to Theorem 1.

THEOREM 8. *Let $\partial: \mathscr{L} \mapsto \mathscr{M}$ be symmetric, weakly additive, and lower extremal and let $f \in \mathscr{L}$ be such that* $\operatorname{dom}(\partial f) = \mathbb{R}^n$. *Then f is both ∂-semismooth and ∂-regular at $x \in \mathbb{R}^n$ if and only if ∂f is both WDC and submonotone at x.*

Proof. Assume that f is ∂-semismooth and ∂-regular at $x \in \mathbb{R}^n$. Then the lim inf in (4), where $T = \partial f$ is equal to the limit of a convergent sequence $\langle v_i - u_i, x_i - x\rangle / |x_i - x|$, where $x_i \to x$, $v_i \in \partial f(x_i)$, and $u_i \in \partial f(x)$. We may assume that $x_i \xrightarrow[h]{} x$. Thus this limit equals

$$\lim_{i \to \infty} \langle v_i - u_i, h\rangle. \tag{6}$$

Since $\partial f(x)$ is compact, we may assume that $u_i \to u \in \partial f(x)$. By ∂-semismoothness, ∂-regularity, and (6), we have

$$\lim_{i \to \infty} \langle v_i, h\rangle = f'(x; h) = \psi^*(h \mid \partial f(x)) \geqslant \langle u, h\rangle.$$

Hence (6) is nonnegative and so ∂f is submonotone at x. Finally, since ∂f is semismooth at x and ∂ is lower extremal, we have that

$$\begin{aligned} \limsup_{y \xrightarrow[h]{} x} \psi^*(h \mid \partial f(y)) &= \limsup_{\substack{y \xrightarrow[h]{} x \\ v \in \partial f(y)}} \langle v, h\rangle \\ &= f'(x; h) \\ &\leqslant \psi^*(h \mid \partial f(x)). \end{aligned}$$

Conversely, we now assume that ∂f is submonotone and WDC at $x \in \mathbb{R}^n$. Suppose $x_i \xrightarrow[h]{} x$ for some $h \in \mathbb{R}^n$, and choose $v_i \in \partial f(x_i)$ for each $i = 1, 2, \ldots$. Now, since ∂f is WDC at x, we have that

$$\limsup_{i \to \infty} \langle v_i, h\rangle \leqslant \psi^*(h \mid \partial f(x)). \tag{7}$$

Let $u \in \partial f(x)$ be such that $\langle u, h\rangle = \psi^*(h \mid \partial f(x))$. Then, by the submonotonicity and local boundedness of ∂f at x, we have

$$\liminf_{i \to \infty} \langle v_i, h\rangle \geqslant \psi^*(h \mid \partial f(x)).$$

This combined with (7) yields the ∂-semismoothness of f at x.

Now, for every $t > 0$,

$$[f(x + th) - f(x)]/t = \langle v, h\rangle$$

for some $v_t \in \overline{\text{co}}\, \partial f(z_t)$ with $z_t \in [x, x + th]$, by Theorem 2. Thus, for each $t > 0$, Caratheodory's Theorem implies the existence of vectors v_{ti}, $i = 1$,

$2, \ldots, n+1$ in $\partial f(z_t)$ and nonnegative scalars λ_{ti}, $i=1, 2, \ldots, n+1$ with $\sum_{i=1}^{n+1} \lambda_{ti} = 1$ such that $\sum_{i=1}^{n+1} \lambda_{ti} v_{ti} = v_t$. Next observe that every sequence $\{t_j\}$ such that $t_j \downarrow 0$ has the property that every one of its subsequences has a further subsequence, say $J \subset \mathbb{N}$, such that the subsequence $\{(\lambda_{t_j 1}, \lambda_{t_j 2}, \ldots, \lambda_{t_j(n+1)}) : j \in J\}$ is convergent to some vector $(\lambda_1, \lambda_2, \ldots, \lambda_{(n+1)})$ satisfying $\sum_{i=1}^{n+1} \lambda_i = 1$ and $\lambda_i \geqslant 0$ for $i = 1, 2, \ldots$. Hence, since f is ∂-semismooth,

$$\lim_J \, [f(x + t_j h) - f(x)]/t_j = \psi^*(h \mid \partial f(x)).$$

Since this is true for every such sequence, we find that $f'(x; h)$ exists with $f'(x; h) = \psi^*(h \mid \partial f(x))$ which establishes the result. ∎

Since both ∂° and $\partial^{\diamond}$ map $\mathscr{L}$ into $\tilde{\mathscr{M}}$, we can combine Proposition 7 and Theorem 8 to obtain the following corollary.

COROLLARY 9. *Let $\partial: \mathscr{L} \mapsto \mathscr{M}$ be symmetric, weakly additive, and lower extremal and suppose that $f \in \mathscr{L}$ is both ∂-semismooth and ∂-regular. If ∂f is convex-valued, then it is maximal submonotone. In particular, this is the case if ∂ is the Clarke or Michel–Penot subdifferential.*

We note that the function f given in Example 4 is both $\partial^{\diamond}$-regular and $\partial^{\diamond}$-semismooth but is not ∂°-regular.

4. The Construction of Other WDC Generalized Subdifferentials

Given $\partial: \mathscr{L} \mapsto \mathscr{M}$, if ∂f is WDC for every $f \in \mathscr{L}$, then we say that ∂ is WDC. The Clarke subdifferential, ∂°, is WDC since $\partial^{\circ} f$ is closed for every $f \in \mathscr{L}$. In this section we construct two other subdifferentials that are WDC. They are always contained in the Clarke subdifferential. Moreover, there are examples for which this containment is strict.

Let $f \in \mathscr{L}$. As is often done, we construct a subdifferential for f by first constructing its associated support functional. In order to obtain the WDC property, we begin by defining

$$f^{\ominus}(x; h) := \limsup_{\substack{y \xrightarrow[h]{} x \\ y \in D}} f'(y; h), \tag{8}$$

where

$$D := \{x \in \mathbb{R}^n : f'(x) \text{ exists}\}.$$

By Rademacher's Theorem, the complement of D has zero measure and so $f^{\ominus}$ is well defined. Consequently, one can apply Caratheodory's Theorem along with (2) and (3) to obtain

$$\begin{aligned} f^{\ominus}(x;h) &= \limsup_{y \xrightarrow[h]{} x,\, u \in \partial^{\circ} f(y)} \langle u, h \rangle \\ &= \limsup_{\substack{y \xrightarrow[h]{} x,\, u \in \partial_2 f(y) \\ y \in \mathrm{dom}(\partial_2 f)}} \langle u, h \rangle \\ &= \limsup_{\substack{y \xrightarrow[h]{} x,\, u \in \partial^{-} f(y) \\ y \in \mathrm{dom}(\partial^{-} f)}} \langle u, h \rangle. \end{aligned} \tag{9}$$

Thus, by the Lebourg Mean Value Theorem, one has

$$f^{-}(x;h) \leqslant f^{\ominus}(x;h) \leqslant f^{\circ}(x;h)$$

for all $x, h \in \mathbb{R}^n$. Moreover, it follows immediately from the definition that

$$\limsup_{y \xrightarrow[h]{} x} f^{\ominus}(y;h) \leqslant f^{\ominus}(x;h) \leqslant f^{\circ}(x;h).$$

Consequently, if we define

$$\partial^{\ominus} f(x) := \{u \in \mathbb{R}^n : \langle u, h \rangle \leqslant f^{\ominus}(x;h)\ \forall h \in \mathbb{R}^n\}, \tag{10}$$

then

$$\partial^{-} f(x) \subset \partial^{\ominus} f(x) \subset \partial^{\circ} f(x)$$

for all $x \in \mathbb{R}^n$. The subdifferential $\partial^{\ominus} f$ has several important properties. It is positively homogeneous, weakly additive, lower extremal, and, for every $f \in \mathscr{L}$, $\partial^{\ominus} f$ is locally bounded. Moreover, $\partial^{\ominus} f$ is WDC. On the other hand, $\partial^{\ominus} f$ is somewhat weak since $f^{\ominus} f(x;\cdot)$ is not necessarily subadditive. Thus, even though $\mathrm{dom}(\partial^{\ominus} f)$ is dense in $\mathbb{R}^n$, it may be empty for some $x \in \mathbb{R}^n$. For this reason we define

$$f^{\oslash}(x;h) := \sup_{k \in \mathbb{R}^n} \{f^{\ominus}(x;h+k) - f^{\ominus}(x;k)\}. \tag{11}$$

The function so obtained is sublinear. The operation employed to obtain $f^{\oslash}(x;\cdot)$ is well known and goes by various names in the literature, e.g., *deconvolution* (for references and related results see Ioffe [10] and Frankowska [8]). This is a standard trick by which a positively homogeneous function can be transformed into a sublinear function. Since $f^{\oslash}(x;\cdot)$ is sublinear, it is the support function of the set

$$\partial^{\oslash} f(x) := \{u \in \mathbb{R}^n : \langle u, h \rangle \leqslant f^{\oslash}(x;h)\ \forall h \in \mathbb{R}^n\} \tag{12}$$

(Hörmander [9]). Unfortunately, the subdifferential $\partial^{\oslash}$ defined in this way is difficult to work with. In particular, we have not been able to determine whether or not $\partial^{\ominus}f$ is locally bounded for all $f \in \mathscr{L}$. In order to compensate for this deficiency, we define

$$f^{\oplus}(x; h) := \inf_{k \in \mathbb{R}^n} \{f^{\oslash}(x; h-k) + f^{\circ}(x; h)\}. \tag{13}$$

Since this is the infimal convolution of two sublinear functions, it is itself sublinear. Consequently, it is the support function for the set

$$\partial^{\oplus}f(x) := \{u \in \mathbb{R}^n : \langle u, h\rangle \leqslant f^{\oplus}(x; h)\ \forall h \in \mathbb{R}^n\}. \tag{14}$$

Clearly,

$$f^{\oplus}(x; \cdot) \leqslant f^{\circ}(x; \cdot), \tag{15}$$

and so

$$\partial^{\oplus}f(x) \subset \partial^{\circ}f(x). \tag{16}$$

Thus, in particular, $\partial^{\oplus}f$ is compact-valued and locally bounded. The primary features of the subdifferential $\partial^{\oplus}f$ are outlined in the following result.

THEOREM 10. *The subdifferential $\partial^{\oplus}: \mathscr{L} \mapsto \mathscr{M}$ defined by Eq. (14) is positively homogeneous, symmetric, weakly additive, lower extremal, and WDC. For each $f \in \mathscr{L}$, $\partial^{\oplus}f$ is compact- and convex-valued with $\operatorname{dom}(\partial^{\oplus}f) = \mathbb{R}^n$, and $\partial^{\oplus}f(x) \subset \partial^{\circ}f(x)$ for all $x \in \mathbb{R}^n$. Furthermore, $\partial^{\oplus}f$ coincides with the Michel–Penot subdifferential whenever the Michel–Penot subdifferential is WDC and submonotone.*

Proof. Let $f \in \mathscr{L}$. The fact that $\partial^{\oplus}f$ is compact- and convex-valued with $\operatorname{dom}(\partial^{\oplus}f) = \mathbb{R}^n$ and the inclusion (16) holding follows immediately from observations that have already been made. The positive homogeneity of $\partial^{\oplus}$ again follows trivially from the definitions. Next, given $x, h \in \mathbb{R}^n$, observe that $(-f)^{\oslash}(x; h) = f^{\ominus}(x; -h)$ and

$$\begin{aligned}(-f)^{\oslash}(x; h) &= \sup_{k \in \mathbb{R}^n} \{(-f)^{\ominus}(x; h+k) - (-f)^{\ominus}(x; k)\} \\ &= \sup_{k \in \mathbb{R}^n} \{f^{\ominus}(x; -h-k) - f^{\ominus}(x; -k)\} \\ &= \sup_{k \in \mathbb{R}^n} \{f^{\ominus}(x; -h+k) - f^{\ominus}(x; k)\}.\end{aligned}$$

Hence

$$\begin{aligned}(-f)^{\oplus}(x; h) &= \inf_{k \in \mathbb{R}^n} \{(-f)^{\oslash}(x; h-k) - (-f)^{\circ}(x; k)\} \\ &= \inf_{k \in \mathbb{R}^n} \{f^{\oslash}(x; -h-k) - f^{\circ}(x; k)\},\end{aligned}$$

and so $\partial^{\oplus}f$ is symmetric.

Now, given x, z, and $h \in \mathbb{R}^n$, note that $(f + \langle z, \cdot \rangle)^{\ominus}(x; h) = f^{\ominus}(x; h) + \langle z, h \rangle$. Hence

$$(f + \langle z, \cdot \rangle)^{\oslash}(x; h) = \sup_{k \in \mathbb{R}^n} \{f^{\ominus}(x; h+k) - f^{\ominus}(x; k) + \langle z, h \rangle\}$$
$$= f^{\ominus}(x; h) + \langle z, h \rangle$$

and so

$$(f + \langle z, \cdot \rangle)^{\oplus}(x; h) = \inf_{k \in \mathbb{R}^n} \{f^{\oslash}(x; h-k) + f^{\circ}(x; k) + \langle z, h \rangle\}$$
$$= f^{\oplus}(x; h) + \langle z, h \rangle.$$

Therefore, $\partial^{\oplus}$ is weakly additive.

We now show that $\partial^{\oplus} f$ is WDC. To this end note that, for every $h \in \mathbb{R}^n$,

$$\begin{aligned} f^{\oplus}(x; h) &= \inf_{k \in \mathbb{R}^n} \{f^{\oslash}(x; h-k) + f^{\circ}(x; k)\} \\ &\geqslant \inf_{k \in \mathbb{R}^n} \{f^{\ominus}(x; h) - f^{\ominus}(x; k) + f^{\circ}(x; k)\} \\ &= f^{\ominus}(x; h) + \inf_{k \in \mathbb{R}^n} \{f^{\circ}(x; k) - f^{\ominus}(x; k)\} \\ &\geqslant f^{\ominus}(x; h), \end{aligned} \tag{17}$$

where the first inequality follows from (11) and the second inequality holds since the expression within the inf operation is always nonnegative. Combining (17) with (9) and (15), we obtain

$$\begin{aligned} \limsup_{y \xrightarrow[h]{} x} f^{\oplus}(y; h) &\leqslant \limsup_{y \xrightarrow[h]{} x} f^{\circ}(y; h) \\ &= f^{\ominus}(x; h) \\ &\leqslant f^{\oplus}(x; h). \end{aligned}$$

Therefore, $\partial^{\oplus} f$ is WDC.

Finally, by applying the Lebourg Mean Value Theorem for the Clarke subdifferential to the expression for $f^{-}(x; h)$, we find that $\partial^{\oplus} f$ is lower extremal from (9) and (17).

Let us now assume that $\partial^{\diamond} f$ is WDC and submonotone. Then, by Theorem 8 and (8), we have

$$f^{\ominus}(x; h) = f'(x; h) = f^{\diamond}(x; h).$$

By (11),

$$f^{\oslash}(x; h) = \sup_{k \in \mathbb{R}^n} \{f'(x; h+k) - f'(x; k)\} = f^{\diamond}(x; h).$$

Consequently,

$$f^{\diamond}(x;h) \geqslant f^{\oplus}(x;h) \geqslant f^{\ominus}(x;h) = f^{\diamond}(x;h),$$

where the first inequality follows from (13) and the second inequality follows from (17). This concludes the proof of the theorem. ∎

We conclude by observing that $\partial^{\oplus} f$ coincides with $\partial^{\diamond} f$ at the origin in Spingarn's Example, but it is strictly contained in $\partial^{\circ} f$ at this point.

Remark. The properties of the subdifferential $\partial^{\oslash}$ have not been discussed. We have avoided doing so since we have not been able to establish whether or not $\partial^{\oslash} f$ is locally bounded. In particular, we do not know whether $\partial^{\oslash} f$ is contained in $\partial^{\circ} f$. Clearly, if this were the case, then the operation (13) used to construct $\partial^{\oplus}$ would be redundant.

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