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Complexity of a Noninterior Path-Following Method for the Linear Complementarity Problem[1]

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Abstract. We study the complexity of a noninterior path-following method for the linear complementarity problem. The method is based on the Chen–Harker–Kanzow–Smale smoothing function. It is assumed that the matrix M is either a P-matrix or symmetric and positive definite. When M is a P-matrix, it is shown that the algorithm finds a solution satisfying the conditions $Mx - y + q = 0$ and $\|\min\{x, y\}\|_\infty \leq \epsilon$ in at most

$$\mathcal{O}((2+\beta)(1+(1/l(M)))^2 \log((1+(1/2)\beta)\mu_0)/\epsilon))$$

Newton iterations; here, β and μ_0 depend on the initial point, $l(M)$ depends on M, and $\epsilon > 0$. When M is symmetric and positive definite, the complexity bound is

$$\mathcal{O}((2+\beta)C^2 \log((1+(1/2)\beta)\mu_0)/\epsilon),$$

where

$$C = 1 + (\sqrt{n}/(\min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\}),$$

and $\lambda_{\min}(M)$, $\lambda_{\max}(M)$ are the smallest and largest eigenvalues of M.

Key Words. Linear complementarity, noninterior path-following methods, complexity of algorithms.

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1. Introduction

We consider a path-following algorithm for solving the following linear complementarity problem:

(LCP(q, M)) find $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$Mx - y + q = 0, \tag{1}$$

$$x \geq 0, \qquad y \geq 0, \qquad x^T y = 0, \tag{2}$$

where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$.

A number of noninterior path following algorithms have recently been proposed that are globally convergent or globally linearly convergent and possess rapid local convergence properties (Refs. 1–16). The complexity of the algorithms has been studied in Xu and Burke (Ref. 17) and in Hotta, Inaba, and Yoshise (Ref. 18). In Ref. 17, the authors established a polynomial complexity bound for an interior-point method based on the Chen–Harker–Kanzow–Smale smoothing function. In Ref. 18, the authors obtain a complexity bound $\mathcal{O}((\bar{\gamma}^6 n/\epsilon^6)\log(\bar{\gamma}^2 n/\epsilon^2))$ for monotone LCP, where $\bar{\gamma}$ is a number which depends on the problem and the initial point. The goal of this paper is to obtain complexity bounds for a noninterior path-following algorithm when the underlying matrix is a P-matrix. In this case, our complexity bound is

$$\mathcal{O}((2+\beta)(1+(1/l(M)))^2 \log((1+2\beta)\mu_0/\epsilon)),$$

where β and μ_0 depend on initial point and $l(M)$ is a fundamental quantity associated with the matrix M. A shortcoming of this bound is that it does not reveal the dependence on the dimension of the problem. In order to obtain a better understanding of this dependence, we consider also the very special case when the matrix M is symmetric and positive definite. This case corresponds precisely to the problem of minimizing a strongly convex quadratic function over the positive orthant. Under this assumption, we obtain the complexity bound

$$\mathcal{O}((2+\beta)C^2 \log((1+2\beta)\mu_0/\epsilon)),$$

with

$$C = 1 + \sqrt{n}/\min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\},$$

where $\lambda_{\min}(M)$, $\lambda_{\max}(M)$ are the smallest and largest eigenvalues of M.

The path-following method considered in this paper is based on the Chen–Harker–Kanzow–Smale smoothing technique (Refs. 5, 11, 19) and as such it relies on the function

$$\phi_\mu(a, b) = a + b - \sqrt{(a-b)^2 + 4\mu^2}. \tag{3}$$

This function is a member of the Chen–Mangasarian class of smoothing functions for the problem LCP(q, M) (Ref. 20). It is easily verified that, for $\mu \geq 0$,

$$\phi_\mu(a, b) = 0 \qquad \text{if and only if } 0 \leq a,\ 0 \leq b,\ ab = \mu^2. \tag{4}$$

The central idea is to apply the Newton method to solve the equation

$$F_{\phi_\mu}(x, y) = 0$$

for decreasing values of μ, where

$$F_{\phi_\mu}(x, y) := \begin{bmatrix} Mx - y + q \\ \Phi_\mu(x, y) \end{bmatrix}, \tag{5}$$

with

$$\Phi_\mu(x, y) := \begin{bmatrix} \phi_\mu(x_1, y_1) \\ \vdots \\ \phi_\mu(x_n, y_n) \end{bmatrix}. \tag{6}$$

The pattern of proof for the Chen–Harker–Kanzow–Smale smoothing function should extend to the smoothed Fischer–Burmeister function,

$$\psi_\mu(a, b) = a + b - \sqrt{a^2 + b^2 + 2\mu^2}, \tag{7a}$$

with

$$\Psi_\mu(x, y) = \begin{bmatrix} \psi_\mu(x_1, y_1) \\ \vdots \\ \psi_\mu(x_n, y_n) \end{bmatrix}. \tag{7b}$$

The plan of this paper is as follows. In Section 2, we establish a global error bound for the Chen–Harker–Kanzow–Smale smoothing function and the smoothed Fischer–Burmeister function in terms of a certain natural residual. This bound is useful in designing the stopping criterion for noninterior path-following methods. We propose a noninterior path-following method in Section 3 and prove its global linear convergence in Section 4. A complexity bound is established for P-matrices in Section 5 and for symmetric positive-definite matrices in Section 6.

2. Global Error Bound

In the following lemma, we bound the quantity $|\min\{a, b\}|$ by a linear function of μ and $|\phi_\mu(a, b)|$. This bound justifies the stopping criterion used in our noninterior path-following methods.

Lemma 2.1. For any $a, b \in \mathbb{R}$ and $\mu \geq 0$, we have

$$|\min\{a, b\}| \leq \mu + (1/2)|\phi_\mu(a, b)|. \tag{8}$$

Proof. Since

$$\begin{aligned} &|2\min\{a, b\} - \phi_\mu(a, b)| \\ &= |a + b - \sqrt{(a-b)^2} - [a + b - \sqrt{(a-b)^2 + 4\mu^2}]| \\ &= \sqrt{(a-b)^2 + 4\mu^2} - \sqrt{(a-b)^2} \\ &\leq \sqrt{(a-b)^2} + 2\mu - \sqrt{(a-b)^2} \\ &= 2\mu, \end{aligned}$$

the result follows. □

Corollary 2.1. For all $\mu \geq 0$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\|\min\{x, y\}\|_\infty \leq \mu + (1/2)\|\Phi_\mu(x, y)\|_\infty, \tag{9}$$

where the mapping $\min\{x, y\}$ is given componentwise by

$$(\min\{x, y\})_i = \min\{x_i, y_i\}, \qquad \text{for } i = 1, 2, \ldots, n.$$

Similar bounds can be established for the Fischer–Burmeister function.

Lemma 2.2. For any $a, b \in \mathbb{R}$ and $\mu \geq 0$, we have

$$|\min\{a, b\}| \leq \mu + 2|\psi_\mu(a, b)|. \tag{10}$$

Proof. Without loss of generality, we assume that $b \geq a$. Then,

$$\min\{a, b\} = a.$$

We can further assume that $a \neq 0$, since in this case (10) trivially holds.

In case $a + b > 0$, we have $b > |a| > 0$ and

$$\begin{aligned} |\psi_\mu(a, b)| &= |a + b - \sqrt{a^2 + b^2 + 2\mu^2}| \\ &= |(a+b)^2 - a^2 - b^2 - 2\mu^2|/(a + b + \sqrt{a^2 + b^2 + 2\mu^2}) \end{aligned}$$

$$
\begin{aligned}
&= 2|ab - \mu^2|/(a + b + \sqrt{a^2 + b^2 + 2\mu^2}) \\
&= 2|a - (\mu^2/b)|/[(a/b) + 1 + \sqrt{(a/b)^2 + 1 + 2(\mu/b)^2}] \\
&\geq 2|a - (\mu^2/b)|/[2 + \sqrt{2 + 2(\mu/b)^2}].
\end{aligned}
$$

So,

$$
\begin{aligned}
2|a| - 2(\mu^2/b) &\leq 2|a - (\mu^2/b)| \\
&\leq [2 + \sqrt{2 + 2(\mu/b)^2}]|\psi_\mu(a, b)|,
\end{aligned}
$$

and

$$
\begin{aligned}
|a| &\leq (\mu^2/b) + (1/2)\,[2 + \sqrt{2 + 2(\mu/b)^2}]|\psi_\mu(a, b)| \\
&\leq (\mu^2/|a|) + (1/2)\,[2 + \sqrt{2 + 2(\mu/a)^2}]|\psi_\mu(a, b)|. \qquad (11)
\end{aligned}
$$

We claim that

$$|\min\{a, b\}| = |a| \leq \mu + 2|\psi_\mu(a, b)|. \qquad (12)$$

Indeed, if $(\mu/a)^2 \leq 1$, then (12) follows from (11); and if $(\mu/a)^2 \geq 1$, then

$$|\min\{a, b\}| = |a| \leq \mu.$$

In case $a + b \leq 0$, we have $a \leq -|b|$ and

$$
\begin{aligned}
|\psi_\mu(a, b)| &= |a + b - \sqrt{a^2 + b^2 + 2\mu^2}| \\
&= \sqrt{a^2 + b^2 + 2\mu^2} - (a + b) \\
&\geq \sqrt{a^2 + b^2 + 2\mu^2} \\
&\geq |a|.
\end{aligned}
$$

So,

$$|\min\{a, b\}| = |a| \leq |\psi_\mu(a, b)|. \qquad (13)$$

The inequality (10) then follows from (12)–(13). □

Corollary 2.2. For all $\mu \geq 0$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\|\min\{x, y\}\|_\infty \leq \mu + 2\|\Psi_\mu(x, y)\|_\infty. \qquad (14)$$

Remark 2.1. In the noninterior path-following algorithm to be studied in Section 3, a sequence of iterates $\{(x^k, y^k, \mu_k)\}$ is generated with

$$y^k = Mx^k + q, \qquad \mu_k \to 0, \qquad \|\Phi_{\mu_k}(x^k, y^k)\|_\infty \to 0.$$

Therefore, by Corollary 2.1, the algorithm reduces the residual $\|\min\{x^k, Mx^k+q\}\|_\infty$ to zero.

Remark 2.2. It is well-known that $\|\min\{x, Mx+q\}\|_\infty$ is a global error bound for LCP(q, M) when M is an R_0-matrix (Refs. 21–22). In particular, this is the case when M is a P-matrix (Ref. 23) or a positive-definite matrix. This implies that there exists a constant c, independent of μ, such that

$$\min_{(x^*, y^*)\in S} \|(x, y)-(x^*, y^*)\|_\infty \le c\,[\mu+(1/2)\|\Phi_\mu(x, y)\|_\infty],$$

$$\min_{(x^*, y^*)\in S} \|(x, y)-(x^*, y^*)\|_\infty \le c\,[\mu+2\|\Psi_\mu(x, y)\|_\infty],$$

where $S(q, M)$ is the solution set to LCP(q, M); that is,

$$S(q, M)=\{(x, y): Mx-y+q=0,\ x^Ty=0,\ x\ge 0,\ y\ge 0\}.$$

Remark 2.3. When $\mu=0$, a similar result was established by Tseng (Ref. 24) for the function ψ_μ. In addition, an Associate Editor has observed that a similar bound can be obtained for the Chen–Mangasarian class (Ref. 20) using results from Refs. 11 and 14.

3. Noninterior Path-Following Algorithm

For $\beta>0$ and $\mu>0$, we define a slice of the neighborhood by

$$\mathcal{N}(\beta, \mu)=\{(x, y): Mx-y+q=0,\ \|\Phi_\mu(x, y)\|_\infty\le\beta\mu\}, \tag{15}$$

and take as our neighborhood of the central path the union of all slices over $\mu>0$, i.e.,

$$\mathcal{N}(\beta)=\bigcup_{\mu>0}\mathcal{N}(\beta, \mu).$$

Algorithm 3.1.

Step 0. Initialization. Let $\mu_0>0$, $\beta>0$, and $(x^0, y^0)\in\mathbb{R}^{2n}$ be given, so that $(x^0, y^0)\in\mathcal{N}(\beta, \mu_0)$, and choose $\sigma_i\in(0, 1]$ and $\alpha_i\in(0, 1)$ for $i=1, 2$.

Step 1. Computation of the Newton Direction. Let $(\Delta x^k, \Delta y^k)$ solve the equation

$$F_{\phi_{\mu_k}}(x^k, y^k)+\nabla F_{\phi_{\mu_k}}(x^k, y^k)^T\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}=0. \tag{16}$$

Step 2. Backtracking Line Search. If $\Phi_{\mu_k}(x^k, y^k) = 0$, set $(x^{k+1}, y^{k+1}) = (x^k, y^k)$; otherwise, let λ_k be the maximum of the values $1, \alpha_1, \alpha_1^2, \ldots,$ such that

$$\|\Phi_{\mu_k}(x^k + \lambda_k \Delta x^k, y^k + \lambda_k \Delta y^k)\|_\infty \leq (1 - \sigma_1 \lambda_k)\|\Phi_{\mu_k}(x^k, y^k)\|_\infty; \tag{17}$$

set

$$(x^{k+1}, y^{k+1}) = (x^k + \lambda_k \Delta x^k, y^k + \lambda_k \Delta y^k).$$

Step 3. Update the Continuation Parameter. Let γ_k be the maximum of the values $1, \alpha_2, \alpha_2^2, \ldots,$ such that

$$\|\Phi_{(1-\sigma_2\gamma_k)\mu_k}(x^{k+1}, y^{k+1})\|_\infty \leq \beta(1 - \sigma_2 \gamma_k)\mu_k; \tag{18}$$

set $\mu_{k+1} = (1 - \sigma_2 \gamma_k)\mu_k$, $k := k + 1$, and return to Step 1.

Remark 3.1. The complexity bound that we obtain in Section 4 depends on both β and μ_0. From the initialization step, we see that these values depend indirectly on M and q. Therefore, understanding the complexity bound requires some understanding of this relationship. In general, the values of β and μ_0 are inversely related. For example, consider the initial point $(x^0, y^0) = (0, q)$. In this case,

$$\begin{aligned}\|\Phi_\mu(x^0, y^0)\|_\infty &= \max\{\sqrt{q_i^2 + 4\mu_0^2} - q_i \mid i = 1, \ldots, n\}\\ &\leq 2(\|q\|_\infty + \mu_0).\end{aligned}$$

Therefore, the conditions in the initialization step are satisfied if

$$\|q\|_\infty \leq (\beta/2 - 1)\,\mu_0. \tag{19}$$

Thus, for example, if we take $\beta = 4$, then we should set $\mu_0 = \|q\|_\infty$, while if $\beta = 2(\|q\|_\infty + 1)$, then we can take $\mu_0 = 1$. In general, we should set

$$\beta = 2(\mu_0^{-1}\|q\|_\infty + 1).$$

We now show that the algorithm is well-defined. For this, we make use of the following lemmas.

Lemma 3.1. See Ref. 12, Lemma 1.2. For any $a, b \in \mathbb{R}^n$ and $\mu > 0$, one has

$$\|\nabla^2 \phi_\mu(a, b)\| \leq 2/\mu.$$

Lemma 3.2. See Ref. 1. For any $\mu_1 \geq 0, \mu_2 \geq 0$, and $a, b \in \mathbb{R}$, we have

$$|\phi_{\mu_1}(a,b) - \phi_{\mu_2}(a,b)| \leq 2|\mu_1 - \mu_2|.$$

Lemma 3.3. Let $0 \leq \lambda \leq 1$, and let $(\Delta x, \Delta y)$ be the solution to the equation

$$F_{\phi_\mu}(x,y) + \nabla F_{\phi_\mu}(x,y)^T \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = 0.$$

Then,

$$\begin{aligned} &\|\Phi_\mu(x + \lambda\Delta x, y + \lambda\Delta y)\|_\infty \\ &\leq (1-\lambda)\|\Phi_\mu(x,y)\|_\infty + (2\lambda^2/\mu)\|(\Delta x, \Delta y)\|_\infty^2. \end{aligned}$$

Proof. By the Taylor expansion, we have that, for $i \in \{1, \dots, n\}$,

$$\begin{aligned} &\phi_\mu(x_i + \lambda\Delta x_i, y_i + \lambda\Delta y_i) \\ &= \phi_\mu(x_i, y_i) + \lambda(\nabla\phi_\mu(x_i, y_i))^T \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} \\ &\quad + (\lambda^2/2)(\Delta x_i, \Delta y_i)\nabla^2\phi_\mu(\bar{x}_i, \bar{y}_i) \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} \\ &= (1-\lambda)\phi_\mu(x_i, y_i) + (\lambda^2/2)(\Delta x_i, \Delta y_i)\nabla^2\phi_\mu(\bar{x}_i, \bar{y}_i) \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix}, \end{aligned}$$

where

$$\bar{x}_i = x_i + \bar{\lambda}\Delta x_i, \bar{y}_i = y_i + \bar{\lambda}\Delta y_i, \qquad 0 \leq \bar{\lambda} \leq 1.$$

Using Lemma 3.1, we have

$$\begin{aligned} &|\phi_\mu(x_i + \lambda\Delta x_i, y_i + \lambda\Delta y_i)| \\ &\leq (1-\lambda)|\phi_\mu(x_i, y_i)| + (\lambda^2/2)\left|(\Delta x_i, \Delta y_i)\nabla^2\phi_\mu(\bar{x}_i, \bar{y}_i) \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix}\right| \\ &\leq (1-\lambda)|\phi_\mu(x_i, y_i)| + (\lambda^2/2)\|(\Delta x_i, \Delta y_i)\| \cdot \|\nabla^2\phi_\mu(\bar{x}_i, \bar{y}_i)\| \left\| \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} \right\| \\ &\leq (1-\lambda)|\phi_\mu(x_i, y_i)| + (\lambda^2/\mu)\|(\Delta x_i, \Delta y_i)\|^2 \\ &\leq (1-\lambda)|\phi_\mu(x_i, y_i)| + (\lambda^2/\mu)\|(\Delta x_i, \Delta y_i)\|_\infty^2, \end{aligned}$$

giving the result. $\square$

To show that the algorithm is well-defined and implementable, we make use of the following assumptions:

(A1) M is a P_0-matrix.
(A2) Given $\beta>0$ and $\mu_0>0$, there exists a $K>0$ such that, for all $\mu\in(0,\mu_0]$ and $\bar{x},\bar{y}\in\mathcal{N}(\beta,\mu)$, we have

$$\|\nabla F_{\phi_\mu}(\bar{x},\bar{y})^{-1}\|_\infty\le K. \tag{20}$$

Assumption (A2) is thoroughly discussed in Ref. 2, where it is also used as one of the basic assumptions in the convergence analysis. As explained in Ref. 2, this assumption is closely related to the uniqueness of the solution to LCP(q, M). In our discussion of complexity, we make use of the following equivalent representation for Assumption (A2).

Lemma 3.4. Let $\beta>0$ and $\mu_0>0$ be given. Then, Assumption (A2) is satisfied if and only if there exists a $C>0$ such that, for all $\mu\in(0,\mu_0]$ and $\bar{x}$, $\bar{y}\in\mathcal{N}(\beta,\mu)$, and for any $z\in\mathbb{R}^n$, every solution to the equation

$$\nabla F_{\phi_\mu}(\bar{x},\bar{y})\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}0\\z\end{bmatrix} \tag{21}$$

must satisfy the bound

$$\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\|_\infty\le C\|z\|_\infty. \tag{22}$$

Remark 3.2. In the next section, we assume that the matrix M is a P-matrix and derive an estimate of the constant C appearing in this lemma. This constant forms the basis for our complexity analysis.

Proof of Lemma 3.4. By setting $C=K$, we obtain the forward implication. For the reverse implication, note that Eq. (21) implies that

$$v=Mu \quad \text{and} \quad (\partial_x\Phi_\mu(\bar{x},\bar{y})+\partial_y\Phi_\mu(\bar{x},\bar{y})M)u=z.$$

Therefore, the inequality (22) implies that every solution to the equation

$$(\partial_x\Phi_\mu(\bar{x},\bar{y})+\partial_y\Phi_\mu(\bar{x},\bar{y})M)u=z$$

must satisfy

$$\|u\|_\infty\le C\|z\|_\infty,$$

or equivalently,

$$\|(\partial_x \Phi_\mu(\bar{x}, \bar{y}) + \partial_y \Phi_\mu(\bar{x}, \bar{y}) M)^{-1}\|_\infty \leq C,$$

for all $\mu \in (0, \mu_0]$ and $(\bar{x}, \bar{y}) \in \mathcal{N}(\beta, \mu)$. Therefore, for all $\mu \in (0, \mu_0]$ and $(\bar{x}, \bar{y}) \in \mathcal{N}(\beta, \mu)$, every solution of the equation

$$\nabla F_{\phi_\mu}(\bar{x}, \bar{y}) \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \bar{w} \\ \bar{z} \end{bmatrix}$$

satisfies

$$\|\bar{u}\|_\infty \leq C(\|\bar{z}\|_\infty + 2\|\bar{w}\|_\infty),$$

since

$$\|\partial_y \Phi_\mu(\bar{x}, \bar{y})\|_\infty \leq 2.$$

In turn, this implies

$$\begin{aligned} \|\bar{v}\|_\infty &\leq \|M\|_\infty \|\bar{u}\|_\infty + \|\bar{w}\|_\infty \\ &\leq C\|M\|_\infty \|\bar{z}\|_\infty + (2C\|M\|_\infty + 1)\|\bar{w}\|_\infty. \end{aligned}$$

Therefore,

$$\left\| \begin{matrix} \bar{u} \\ \bar{v} \end{matrix} \right\|_\infty \leq 2[2C \max\{\|M\|_\infty, 1\} + 1] \left\| \begin{matrix} \bar{w} \\ \bar{z} \end{matrix} \right\|_\infty,$$

for all $\mu \in (0, \mu_0]$ and $(\bar{x}, \bar{y}) \in \mathcal{N}(\beta, \mu)$, or equivalently Assumption (A2) holds with

$$K = 2[2C \max\{\|M\|_\infty, 1\} + 1]. \qquad \square$$

We now show that the algorithm is well-defined and implementable.

Theorem 3.1. Let $\beta > 0$, and let $(x^k, y^k) \in \mathcal{N}(\beta, \mu_k)$ for some $\mu_k > 0$. Suppose that Assumptions (A1) and (A2) hold, and let the constant C be as given in Lemma 3.4.

(i) See Ref. 11. The Jacobian $\nabla F_{\phi_{\mu_k}}(x^k, y^k)$ is nonsingular. Hence, the Newton direction in Step 1 of the algorithm exists [see (16)] and is unique.

(ii) If $\Phi_{\mu_k}(x^k, y^k) \neq 0$, then $\lambda_k \geq \bar{\lambda}$, where

$$\bar{\lambda} = \alpha_1 \tilde{\lambda}, \qquad \tilde{\lambda} = \min\{1, (1 - \sigma_1)/2\beta C^2\}. \tag{23}$$

Hence, the backtracking procedure for evaluating λ_k in Step 2 is finitely terminating.

(iii) $\gamma_k \geq \bar{\gamma}$, where

$$\bar{\gamma} = \min\{1, \sigma_2^{-1}\alpha_2\tilde{\gamma}\}, \qquad \tilde{\gamma} = \sigma_1\bar{\lambda}\beta/(2+\beta). \tag{24}$$

Hence, the backtracking procedure for evaluating γ_k in Step 3 is finitely terminating.

Proof.

(i) See Kanzow (Ref. 11).

(ii) Let $(\Delta x^k, \Delta y^k)$ be chosen so as to satisfy the Newton Eq. (16). It follows from Lemma 3.3 and Assumption (A2) that

$$\begin{aligned}
&\|\Phi_{\mu_k}(x^k+\lambda\Delta x^k, y^k+\lambda\Delta y^k)\|_\infty \\
&\leq (1-\lambda)\|\Phi_{\mu_k}(x^k, y^k)\|_\infty + (2\lambda^2/\mu_k)\|(\Delta x^k, \Delta y^k)\|_\infty^2 \\
&\leq (1-\lambda)\|\Phi_{\mu_k}(x^k, y^k)\|_\infty + (2\lambda^2/\mu_k)C^2\|\Phi_{\mu_k}(x^k, y^k)\|_\infty^2 \\
&\leq (1-\lambda)\|\Phi_{\mu_k}(x^k, y^k)\|_\infty + (2/\mu_k)\lambda^2 C^2\beta\mu_k\|\Phi_{\mu_k}(x^k, y^k)\|_\infty \\
&= (1-\lambda)\|\Phi_{\mu_k}(x^k, y^k)\|_\infty + 2\lambda^2 C^2\beta\|\Phi_{\mu_k}(x^k, y^k)\|_\infty \\
&\leq (1-\sigma_1\lambda)\|\Phi_{\mu_k}(x^k, y^k)\|_\infty, \qquad \text{for all } \lambda \in [0, \tilde{\lambda}].
\end{aligned}$$

Therefore, $\lambda_k \geq \bar{\lambda}$ with $\bar{\lambda} = \alpha_1\bar{\lambda}$.

(iii) We consider two cases.

If $\|\Phi_{\mu_k}(x^k, y^k)\|_\infty = 0$, then

$$x^{k+1} = x^k \quad \text{and} \quad y^{k+1} = y^k.$$

Thus, by Lemma 3.2,

$$\begin{aligned}
&\|\Phi_{(1-\gamma)\mu_k}(x^{k+1}, y^{k+1})\|_\infty/(1-\gamma)\mu_k \\
&= \|\Phi_{(1-\gamma)\mu_k}(x^k, y^k)\|_\infty/(1-\gamma)\mu_k \\
&\leq [\|\Phi_{\mu_k}(x^k, y^k)\|_\infty + 2\gamma\mu_k]/(1-\gamma)\mu_k \\
&= 2\gamma\mu_k/(1-\gamma)\mu_k \\
&\leq \beta, \qquad \text{for all } \gamma \in [0, \tilde{\gamma}].
\end{aligned}$$

If $\|\Phi_{\mu_k}(x^k, y^k)\|_\infty \neq 0$, then by part (ii) and Lemma 3.2, we have

$$\begin{aligned}
&\|\Phi_{(1-\gamma)\mu_k}(x^{k+1}, y^{k+1})\|_\infty/(1-\gamma)\mu_k \\
&\leq [\|\Phi_{\mu_k}(x^{k+1}, y^{k+1})\|_\infty + 2\gamma\mu_k]/(1-\gamma)\mu_k \\
&\leq [(1-\sigma_1\bar{\lambda})\|\Phi_{\mu_k}(x^k, y^k)\|_\infty + 2\gamma\mu_k]/(1-\gamma)\mu_k \\
&\leq [(1-\sigma_1\bar{\lambda})\beta\mu_k + 2\gamma\mu_k]/(1-\gamma)\mu_k \\
&= [(1-\sigma_1\bar{\lambda})\beta + 2\gamma]/(1-\gamma) \\
&\leq \beta, \qquad \text{for all } \gamma \in [0, \tilde{\gamma}].
\end{aligned}$$

Therefore,

$$\gamma_k \geq \bar{\gamma}, \qquad \text{with } \bar{\gamma} = \min\{1, \sigma_2^{-1}\alpha_2\tilde{\gamma}\}. \qquad \square$$

We now state and prove the global linear convergence result for the algorithm described in the preceding section.

Theorem 3.2. Suppose that M is a P_0-matrix, Assumption (A2) holds, and the constant C is as given by Lemma 3.4. Let (x^k, y^k, μ_k) be the sequence generated by the algorithm of the preceding section. Then:

(i) For $k = 0, 1, \ldots,$

$$Mx^k - y^k + q = 0, \tag{25}$$

$$(x^k, y^k) \in \mathcal{N}(\beta, \mu_k), \tag{26}$$

$$(1-\sigma_2\gamma_{k-1})\cdots(1-\sigma_2\gamma_0)\mu_0 = \mu_k. \tag{27}$$

(ii) For all $k \geq 0$, we have

$$\gamma_k \geq \bar{\gamma} := \min\{1, \sigma_2^{-1}\alpha_2\sigma_1\bar{\lambda}\beta(2+\beta)^{-1}\}, \tag{28}$$

where

$$\bar{\lambda} = \alpha_1 \min\{1, (1-\sigma_1)/2\beta C^2\}.$$

Therefore, μ_k converges to 0 at a global linear rate.

(iii) The sequence $\{(x^k, y^k)\}$ is bounded and converges to a solution of LCP(q, M).

(iv) Take $C > 1$, let $\epsilon > 0$, and choose $\sigma_2 \in (0, 1]$ close enough to 1 to ensure that

$$\bar{\gamma} = (1/2)\sigma_2^{-1}\alpha_2\sigma_1\alpha_1(1-\sigma_1)(2+\beta)^{-1}C^{-2}.$$

Then, the algorithm locates a solution in the set

$$\{(x, y): Mx - y + q = 0, \|\min\{x, y\}\|_\infty \leq \epsilon\}$$

in

$$\mathcal{O}((2+\beta)C^2 \log \epsilon^{-1}(1+(1/2)\beta)\mu_0) \tag{29}$$

steps.

Proof.

(i) We establish (25)–(27) by induction on k. Clearly, these relations hold for $k=0$. Now, assume that they hold for some $k>0$. By Theorem 3.1, the algorithm is well defined and so (26)–(27) hold with k replaced by $k+1$. Since (16) is satisfied for all k, with

$$Mx^0 - y^0 + q = 0,$$

we have that

$$Mx^k - y^k + q = 0, \qquad \text{for all } k,$$

and so, in particular, it is true when k replaced by $k+1$. Hence, by induction, (25)–(27) hold for all k.

(ii) This follows from Theorem 3.1.

(iii) By Assumption (A2) and part (ii), we have

$$\begin{aligned}
&\|(x^{k+1}, y^{k+1}) - (x^k, y^k)\|_\infty \\
&= \lambda_k \|(\Delta x^k, \Delta y^k)\|_\infty \\
&\le C\|\Phi_{\mu_k}(x^k, y^k)\|_\infty \\
&\le C\beta\mu_k \\
&\le C\beta(1-\sigma_2\bar{\gamma})^k.
\end{aligned}$$

Therefore, $\{(x^k, y^k)\}$ is a Cauchy sequence, which is bounded and converges to a point (x^*, y^*). It follows from $(x^k, y^k) \in \mathcal{N}(\beta, \mu_k)$ that $(x^*, y^*) \in S$.

(iv) It follows from Corollary 2.1 and parts (i) and (ii) that

$$\begin{aligned}
&\|\min\{x^k, Mx^k + q\}\|_\infty \\
&= \|\min\{x^k, y^k\}\|_\infty \\
&\le \mu_k + (1/2)\|\Phi_{\mu_k}(x^k, y^k)\|_\infty \\
&\le [1+(1/2)\beta]\,\mu_k \\
&= [1+(1/2)\beta]\,(1-\sigma_2\gamma_{k-1})\cdots(1-\sigma_2\gamma_0)\,\mu_0 \\
&\le [1+(1/2)\beta]\,(1-\sigma_2\bar{\gamma})^k\mu_0 \\
&= [1+(1/2)\beta]\,\mu_0[1-\delta(2+\beta)^{-1}C^{-2}]^k,
\end{aligned}$$

where

$$\delta = \alpha_1 \alpha_2 \sigma_1 (1 - \sigma_1)/2.$$

To ensure that

$$[1 + (1/2)\beta]\, \mu_0 [1 - \delta(2 + \beta)^{-1} C^{-2}]^k \leq \epsilon,$$

it suffices to have

$$\begin{aligned} &k \log[1 - \delta(2 + \beta)^{-1} C^{-2}] \\ &\leq k(-\delta)\, C^{-2}(2 + \beta)^{-1} \\ &\leq \log[\epsilon\{[1 + (1/2)\beta]\, \mu_0\}^{-1}]. \end{aligned}$$

Therefore,

$$k > \delta^{-1}(2 + \beta) C^2 \log\{\epsilon^{-1}[1 + (1/2)\beta]\, \mu_0\},$$

completing the proof. □

Part (iv) of Theorem 3.2 shows that, if we can bound the constant C by a function of M, then we can obtain a complexity bound in terms of the problem data M, q, β, and μ_0.

4. Complexity Bound

We begin by showing how the constant C in Lemma 3.4 of Section 3 can be bounded by a fundamental quantity associated with P-matrices. This bound is the key to our complexity analysis.

It is well known that a matrix $M \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if, for every $x \in \mathbb{R}^n$ and $x \neq 0$,

$$\max_{1 \leq i \leq n} x_i (Mx)_i > 0.$$

Using this fact, Mathias and Pang (Ref. 23) introduce the quantity

$$l_0(M) := \min_{\|x\|_\infty = 1} \max_{1 \leq i \leq n} x_i (Mx)_i, \tag{30}$$

for P-matrices. They show how this quantity can be used to derive error bounds for the linear complementarity problem. It is easy to see that $l_0(M)$ is well defined, finite, and positive. Moreover, for any $x \in \mathbb{R}^n$,

$$l_0(M) \|x\|_\infty^2 \leq \max_{1 \leq i \leq n} x_i (Mx)_i. \tag{31}$$

We have the following bounds for $l_0(M)$.

Lemma 4.1. Given $M \in \mathbb{R}^{n\times n}$, define

$$\tilde{M} = (1/2)(M + M^T)$$

to be the symmetric part of M and

$$\lambda_{\min}(\tilde{M}) = \min\{\lambda | \lambda \text{ is an eigenvalue of } \tilde{M}\}$$

to be the smallest eigenvalue of $\tilde{M}$. Then,

$$n^{-1}\lambda_{\min}(\tilde{M}) \leq l_0(M) \leq \|M\|_\infty. \tag{32}$$

Proof. For the upper bound we have

$$\begin{aligned} l_0(M) &= \min_{\|x\|_\infty = 1} \max_{i=1,\dots,n} x_i(Mx)_i \\ &\leq \min_{\|x\|_\infty = 1} \|x\|_\infty \|Mx\|_\infty \\ &\leq \min_{\|x\|_\infty = 1} \|x\|_\infty^2 \|M\|_\infty \\ &= \|M\|_\infty. \end{aligned}$$

For the lower bound, we first observe that, when $\lambda_{\min}(\tilde{M}) \leq 0$, the result is trivially true. Therefore, we can assume that M is positive definite, i.e.,

$$\lambda_{\min}(\tilde{M}) > 0.$$

In this case, we have

$$\begin{aligned} l_0(M) &= \min_{\|x\|_\infty = 1} \max_{i=1,\dots,n} x_i(Mx)_i \\ &\geq \min_{\|x\|_\infty = 1} n^{-1} x^T M x \\ &\geq n^{-1} \min_{\|x\|_\infty = 1} (x^T/\|x\|_2) M (x/\|x\|_2) \\ &= n^{-1} \min_{\|x\|_2 = 1} x^T M x \\ &= n^{-1} \min_{\|x\|_2 = 1} x^T \tilde{M} x \\ &= n^{-1}\lambda_{\min}(\tilde{M}), \end{aligned}$$

where the first inequality follows since the maximum is always larger than the average, the second inequality follows since

$$\|x\|_2 \geq \|x\|_\infty = 1,$$

and the second to last equality follows since

$$x^T M x = x^T \tilde{M} x. \qquad \square$$

We also make use of principal pivotal transformations. Let α be a subset of $\{1, \ldots, n\}$ and set $\bar{\alpha} = \{1, \ldots, n\} \backslash \alpha$. By means of a principal rearrangement, we may assume that $M_{\alpha\alpha}$ is a leading principal submatrix of M. The principal pivotal transform of M with respect to the index set α is the matrix

$$\mathscr{P}_\alpha(M) = \begin{bmatrix} M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} & M / M_{\alpha\alpha} \end{bmatrix}, \tag{33}$$

where

$$M / M_{\alpha\alpha} = M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \tag{34}$$

is the Schur complement of $M_{\alpha\alpha}$ in M. The following two properties of the principal pivotal transform $\mathscr{P}_\alpha(M)$ can be found in Cottle, Pang, and Stone (Ref. 25).

Lemma 4.2. See Ref. 25, Section 2.3. Let α be a subset of $\{1, \ldots, n\}$. Suppose that the principal submatrix $M_{\alpha\alpha}$ is nonsingular and

$$\begin{bmatrix} y_\alpha \\ y_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} x_\alpha \\ x_{\bar{\alpha}} \end{bmatrix}. \tag{35}$$

Then,

$$\begin{bmatrix} x_\alpha \\ y_{\bar{\alpha}} \end{bmatrix} = \mathscr{P}_\alpha(M) \begin{bmatrix} y_\alpha \\ x_{\bar{\alpha}} \end{bmatrix}. \tag{36}$$

Lemma 4.3. See Ref. 25, Theorem 4.1.3. Let α be any subset of $\{1, \ldots, n\}$. If M is a P-matrix, then so is $\mathscr{P}_\alpha(M)$.

Define

$$l(M) := \min\{l_0(\mathscr{P}_\alpha(M)) \mid \alpha \subseteq \{1, \ldots, n\}\}. \tag{37}$$

Since there are only finitely many principal pivotal transformations of M, we have $l(M) > 0$ and

$$l(M)\|x\|_\infty^2 \le \max_{1 \le i \le n} x_i (P_\alpha(M)x)_i, \quad \text{for all } x \in \mathbb{R}^n \text{ and } \alpha \subseteq \{1, \ldots, n\}. \tag{38}$$

In addition, Lemma 4.1 gives the lower bound

$$l(M) \ge n^{-1} \min\{\lambda_{\min}(\tilde{\mathscr{P}}_\alpha(\tilde{M})) \mid \alpha \subseteq \{1, \ldots, n\}\}. \tag{39}$$

The quantity $l(M)$ can be used to obtain an upper bound for the value of C in Lemma 3.4.

Lemma 4.4. Let $\beta > 0, \mu > 0$, and $\bar{x}, \bar{y} \in \mathcal{N}(\beta, \mu)$ be given. If $z \in \mathbb{R}^n$, then the system

$$\nabla F_{\phi_\mu}(\bar{x}, \bar{y}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ z \end{bmatrix} \tag{40}$$

has a unique solution $[\begin{smallmatrix} \bar{u} \\ \bar{v} \end{smallmatrix}]$, and this solution satisfies

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_\infty \leq [1 + l(M)^{-1}] \|z\|_\infty .$$

Proof. Let $\beta > 0, \mu > 0, \bar{x}, \bar{y} \in \mathcal{N}(\beta, \mu)$, and $z \in \mathbb{R}^n$ be given. By Part (i) of Theorem 3.1, the system (40) has a unique solution $[\begin{smallmatrix} \bar{u} \\ \bar{v} \end{smallmatrix}]$. Set

$$D_1 = \operatorname{diag}[1 - (\bar{x}_i - \bar{y}_i)/\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\mu^2}],$$
$$D_2 = \operatorname{diag}[1 + (\bar{x}_i - \bar{y}_i)/\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\mu^2}].$$

Then,

$$\nabla F_{\phi_\mu}(\bar{x}, \bar{y}) = \begin{bmatrix} M & -I \\ D_1 & D_2 \end{bmatrix}.$$

Let

$$\alpha = \{i | \bar{x}_i - \bar{y}_i \leq 0\},$$

and let

$$\hat{u} = \begin{bmatrix} (\bar{v})_\alpha \\ (\bar{u})_{\bar{\alpha}} \end{bmatrix}, \qquad \hat{v} = \begin{bmatrix} (\bar{u})_\alpha \\ (\bar{v})_{\bar{\alpha}} \end{bmatrix}, \tag{41}$$

$$\hat{x} = \begin{bmatrix} \bar{y}_\alpha \\ \bar{x}_{\bar{\alpha}} \end{bmatrix}, \qquad \hat{y} = \begin{bmatrix} \bar{x}_\alpha \\ \bar{y}_{\bar{\alpha}} \end{bmatrix}. \tag{42}$$

By Lemma 4.2,

$$\mathcal{P}_\alpha(M)\,\hat{u} - \hat{v} = 0, \tag{43}$$
$$\hat{D}_1 \hat{u} + \hat{D}_2 \hat{v} = z, \tag{44}$$

where

$$\hat{D}_1 = \operatorname{diag}[1 - (\hat{x}_i - \hat{y}_i)/\sqrt{(\hat{x}_i - \hat{y}_i)^2 + 4\mu^2}],$$
$$\hat{D}_2 = \operatorname{diag}[1 + (\hat{x}_i - \hat{y}_i)/\sqrt{(\hat{x}_i - \hat{y}_i)^2 + 4\mu^2}].$$

Here, it is important to note that

$$1 + (\hat{x}_i - \hat{y}_i)/\sqrt{(\hat{x}_i - \hat{y}_i)^2 + 4\mu^2} \geq 1, \qquad \text{for } i = 1, \ldots, n. \tag{45}$$

By (43),

$$\hat{v} = \mathscr{P}_\alpha(M)\hat{u}.$$

Substituting this expression for $\hat{v}$ into (44) yields

$$\hat{D}_1 \hat{u} + \hat{D}_2 \mathscr{P}_\alpha(M)\hat{u} = z.$$

Multiplying both sides of this equation by $\hat{D}_2^{-1}$ gives us

$$\hat{D}_2^{-1} \hat{D}_1 \hat{u} + \mathscr{P}_\alpha(M)\, \hat{u} = \hat{D}_2^{-1} z. \tag{46}$$

Therefore,

$$\begin{aligned} l(M)\|\hat{u}\|_\infty^2 &\leq \max_{1 \leq i \leq n} (\hat{u})_i [\mathscr{P}_\alpha(M)\hat{u}]_i \\ &\leq \max_{1 \leq i \leq n} (\hat{u})_i [\hat{D}_2^{-1} \hat{D}_1 \hat{u} + \mathscr{P}_\alpha(M)\hat{u}]_i \\ &\leq \max_{1 \leq i \leq n} |(\hat{u})_i| \max_{1 \leq i \leq n} |(\hat{D}_2^{-1} z)_i| \\ &= \|\hat{u}\|_\infty \|\hat{D}_2^{-1} z\|_\infty \\ &\leq \|\hat{u}\|_\infty \|\hat{D}_2^{-1}\|_\infty \|z\|_\infty \\ &\leq \|\hat{u}\|_\infty \|z\|_\infty, \end{aligned}$$

and so

$$\|\hat{u}\|_\infty \leq l(M)^{-1} \|z\|_\infty. \tag{47}$$

Also, by (44) and (47), we have

$$\begin{aligned} \|\hat{v}\|_\infty &= \|\hat{D}_2^{-1} z - \hat{D}_2^{-1} \hat{D}_1 \hat{u}\|_\infty \\ &\leq \|\hat{D}_2^{-1}\|_\infty \|z\|_\infty + \|\hat{D}_2^{-1}\|_\infty \|\hat{D}_1\|_\infty \|\hat{u}\|_\infty \\ &\leq \|z\|_\infty + \|\hat{u}\|_\infty \\ &\leq [1 + l(M)^{-1}] \|z\|_\infty. \end{aligned} \tag{48}$$

Combining (47)–(48), we obtain

$$\|(\bar{u}, \bar{v})\| = \|(\hat{u}, \hat{v})\|_\infty \leq [1 + l(M)^{-1}] \|z\|_\infty. \qquad \square$$

By Lemma 4.4, one may take $C = [1 + l(M)^{-1}]$ in Lemma 3.4 whenever M is a P-matrix. For simplicity, we assume that $\sigma_2 = 1$ in the algorithm.

Making this substitution in part (iv) of Theorem 3.2 yields the following complexity bound.

Theorem 4.1. Assume that M is a P-matrix. Given $\epsilon > 0$, the algorithm of Section 3 finds a solution in the set

$$\{(x, y): Mx - y + q = 0, \|\min\{x, y\}\|_\infty \leq \epsilon\}$$

via

$$\mathcal{O}((2+\beta)[1 + l(M)^{-1}]^2 \log\{\epsilon^{-1}[1 + (1/2)\beta]\mu_0\}) \tag{49}$$

steps, where $l(M)$ is defined in (37).

By making use of the relation (19) for the initial point $(x^0, y^0) = (0, q)$, we convert this complexity bound into a bound involving M and q alone.

Corollary 4.1. Let the assumptions of Theorem 4.1 hold. In addition, assume that

$$(x^0, y^0) = (0, q), \qquad \mu_0 = \|q\|_\infty \neq 0, \qquad \beta = 4.$$

Then, the complexity bound (49) in Theorem 4.1 becomes

$$\mathcal{O}([1 + l(M)^{-1}]^2 \log(\epsilon^{-1}\|q\|_\infty)). \tag{50}$$

5. Symmetric Positive-Definite Case

A drawback to the complexity bound given in Theorem 4.1 is that it does not reveal the dependence on the dimension n. Since the constant $l(M)$ depends on all of the principal pivot transforms of M, it is possible that this constant grows rapidly with the dimension of M. In order to gain some insight into the relationship between dimension and complexity, we consider a very special instance of LCP(q, M). Our goal here is not necessarily to improve on the bound given in Theorem 4.1, but rather to understand how the constant C in part (iv) of Theorem 3.2 may grow as a function of the dimension of the problem. For this, we assume that the matrix M is symmetric and positive definite. Under this assumption, LCP(q, M) is equivalent to the problem of minimizing a strongly convex quadratic function subject to nonnegativity constraints.

The strategy in this section is to obtain a lower bound on the quantity $\lambda_{\min}(\tilde{\mathscr{P}}_\alpha(\tilde{M}))$ appearing in (39) that is independent of α. This in turn yields a complexity bound that is independent of the principal pivot transforms of M.

Let $\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)$ denote the eigenvalues of M. We adopt the convention that

$$\lambda_{\min}(M) = \lambda_n(M) \leq \cdots \leq \lambda_2(M) \leq \lambda_1(M) = \lambda_{\max}(M).$$

In our analysis, we make use of the Cauchy interlacing theorem for real symmetric matrices and a recent interlacing theorem due to Smith (Ref. 26) for the Schur complement of a real symmetric matrix. Both results are stated below for the reader convenience.

Theorem 5.1. See Ref. 26. Cauchy Interlacing Theorem. Let M be an $n \times n$ symmetric matrix and have the partitioned form

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}, \tag{51}$$

where $M_{\alpha\alpha} \in \mathbb{R}^{r \times r}$. Then,

$$\lambda_{i+n-r}(M) \leq \lambda_i(M_{\alpha\alpha}) \leq \lambda_i(M), \qquad \text{for } i = 1, 2, \ldots, r. \tag{52}$$

Theorem 5.2. See Ref. 26. Let M be an $n \times n$ symmetric positive-semi-definite matrix and have the partitioned form

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}, \tag{53}$$

where $M_{\alpha\alpha} \in \mathbb{R}^{r \times r}$. Then,

$$\lambda_{i+r}(M) \leq \lambda_i(M/M_{\alpha\alpha}) \leq \lambda_i(M), \qquad \text{for } i = 1, 2, \ldots, n-r. \tag{54}$$

Note that, when M is symmetric and α is any subset of $\{1, \ldots, n\}$, the principal pivotal transform $\mathscr{P}_\alpha(M)$ is skew symmetric and

$$\begin{aligned} \tilde{\mathscr{P}}_\alpha(\tilde{M}) &:= (1/2)[\mathscr{P}_\alpha(M) + \mathscr{P}_\alpha(M)^T] \\ &= \begin{bmatrix} M_{\alpha\alpha}^{-1} & 0 \\ 0 & M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \end{bmatrix} \end{aligned} \tag{55}$$

is symmetric. It is easy to see that, if M is symmetric positive definite, then so is $\tilde{\mathscr{P}}_\alpha(\tilde{M})$. Using Theorems 5.1 and 5.2, we establish a lower bound for $\lambda_{\min}(\tilde{\mathscr{P}}_\alpha(\tilde{M}))$ that is independent of α.

Theorem 5.3. Let M be a symmetric positive-definite matrix. Then,

$$\lambda_{\min}(\tilde{\mathscr{P}}_\alpha(\tilde{M})) \geq \min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\}, \tag{56}$$

for any $\alpha \subseteq \{1, \ldots, n\}$.

Proof. It follows from (55) that

$$\lambda_{\min}(\tilde{\mathscr{P}}_\alpha(\tilde{M})) = \min\{\lambda_{\min}(M_{\alpha\alpha}^{-1}), \lambda_{\min}(M/M_{\alpha\alpha})\}.$$

By Theorem 5.1, we have

$$\lambda_{\min}(M_{\alpha\alpha}^{-1}) \geq 1/\lambda_{\max}(M),$$

and by Theorem 5.2, we have

$$\lambda_{\min}(M/M_{\alpha\alpha}) \geq \lambda_{\min}(M).$$

Therefore,

$$\lambda_{\min}(\tilde{\mathscr{P}}_\alpha(\tilde{M})) \geq \min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\}. \qquad \square$$

By combining Theorems 4.1 and 5.3 with relation (39), we obtain immediately a complexity bound of

$$\mathscr{O}((2+\beta)K^2 \log\{\epsilon^{-1}[1+(1/2)\beta]\mu_0\}), \tag{57}$$

where

$$K = 1 + n(\min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\})^{-1}.$$

However, this bound can be improved significantly by using Theorem 5.3 to directly approximate the constant C appearing in Lemma 3.4.

Lemma 5.1. Let $\beta > 0, \mu > 0$, and $\bar{x}, \bar{y} \in \mathscr{N}(\beta, \mu)$ be given. If $z \in \mathbb{R}^n$, then the system

$$\nabla F_{\phi_\mu}(\bar{x}, \bar{y}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ z \end{bmatrix} \tag{58}$$

has a unique solution $[\begin{smallmatrix} \bar{u} \\ \bar{v} \end{smallmatrix}]$ and this solution satisfies

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_\infty \leq (1 + \sqrt{n}[\min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\}]^{-1}) \|z\|_\infty .$$

Proof. The notation used here is the same as that used in Lemma 4.4. From Eq. (46), we have

$$\hat{D}_2^{-1}\hat{D}_1\hat{u} + \mathscr{P}_\alpha(M)\hat{u} = \hat{D}_2^{-1}z. \tag{59}$$

Therefore,

$$\begin{aligned}
&\min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\}\|\hat{u}\|_2^2\\
&\leq \lambda_{\min}(\tilde{\mathscr{P}}_\alpha(\tilde{M}))\|\hat{u}\|_2^2\\
&\leq \hat{u}^T\tilde{\mathscr{P}}_\alpha(\tilde{M})\hat{u}\\
&= \hat{u}^T\tilde{\mathscr{P}}_\alpha(\tilde{M})\hat{u}\\
&\leq \hat{u}^T(\hat{D}_2^{-1}\hat{D}_1 + \mathscr{P}_\alpha(M))\,\hat{u}\\
&\leq |\hat{u}^T\hat{D}_2^{-1}z|\\
&\leq \|\hat{u}\|_2\|\hat{D}_2^{-1}z\|_2\\
&\leq \sqrt{n}\|\hat{u}\|_2\|\hat{D}_2^{-1}z\|_\infty\\
&\leq \sqrt{n}\|\hat{u}\|_2\|\hat{D}_2^{-1}\|_\infty\|z\|_\infty\\
&\leq \sqrt{n}\|\hat{u}\|_2\|z\|_\infty,
\end{aligned}$$

and so,

$$\|\hat{u}\|_\infty \leq \|\hat{u}\|_2 \leq \sqrt{n}\,(\min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\})^{-1}\|z\|_\infty.$$

The remainder of the proof follows the pattern given for Lemma 4.4. □

As in Theorem 4.1, this bound yields the following complexity result.

Theorem 5.4. Assume that M is a symmetric positive-definite matrix. Given $\epsilon > 0$, the algorithm of Section 3 finds a solution in the set

$$\{(x, y): Mx - y + q = 0, \|\min\{x, y\}\|_\infty \leq \epsilon\}$$

via

$$\mathscr{O}((2+\beta)C^2 \log\{\epsilon^{-1}[1 + (1/2)\beta]\mu_0\}) \tag{60}$$

steps, where

$$C = 1 + \sqrt{n}(\min\{\lambda_{\min}(M), 1/\lambda_{\max}(M)\})^{-1},$$

and $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ are the smallest and largest eigenvalues of M.

Although the relation (39) implies immediately that the complexity bound given by (57) is not as sharp as that appearing in Theorem 4.1, the complexity bounds in Theorems 5.4 and 4.1 are not easily compared. Understanding the relationship between these complexity bounds remains an open question.

References

1. Burke, J., and Xu, S., *The Global Linear Convergence of a Noninterior Path-Following Algorithm for Linear Complementarity Problems*, Mathematics of Operations Research, Vol. 23, pp. 719–734, 1998.
2. Burke, J., and Xu, S., *A Noninterior Predictor-Corrector Path-Following Algorithm for the Monotone Linear Complementarity Problem*, Mathematical Programming, Vol. 87, pp. 113–130, 2000.
3. Chen, B., and Chen, X., *A Global Linear and Local Quadratic Continuation Method for Variational Inequalities with Box Constraints*, Computational Optimization and Applications, Vol. 17, pp. 131–158, 2000.
4. Chen, B., and Chen, X., *A Global and Local Superlinear Continuation Method for $P_0 + R_0$ and Monotone NCP*, SIAM Journal on Optimization, Vol. 9, pp. 605–623, 1999.
5. Chen, B., and Harker, P. T., *A Noninterior Point Continuation Method for Linear Complementarity Problems*, SIAM Journal on Matrix Analysis and Applications, Vol. 14, pp. 1168–1190, 1993.
6. Chen, B., and Xiu, N., *A Global Linear and Local Quadratic Noninterior Continuation Method for Nonlinear Complementarity Problems Based on Chen–Mangasarian Smoothing Functions*, SIAM Journal on Optimization, Vol. 9, pp. 605–623, 1999.
7. Chen, X., Qi, L., and Sun, D., *Global and Superlinear Convergence of the Smoothing Newton Method and Its Application to General Box-Constrained Variational Inequalities*, Mathematics of Computation, Vol. 67, pp. 519–540, 1998.
8. Chen, X., and Ye, Y., *On Homotopy-Smoothing Methods for Box-Constrained Variational Inequalities*, SIAM Journal on Control and Optimization, Vol. 37, pp. 589–616, 1999.
9. Hotta, K., and Yoshise, A., *Global Convergence of a Class of Noninterior Point Algorithms Using Chen–Harker–Kanzow Functions for Nonlinear Complementarity Problems*, Mathematical Programming, Vol. 86, pp. 105–133, 1999.
10. Jiang, H., *Global Convergence Analysis of the Generalized Newton and Gauss–Newton Methods for the Fischer–Burmeister Equation for the Complementarity Problem*, Mathematics of Operations Research, Vol. 24, pp. 529–543, 1999.
11. Kanzow, C., *Some Noninterior Continuation Methods for Linear Complementarity Problems*, SIAM Journal on Matrix Analysis and Applications, Vol. 17, pp. 851–868, 1996.
12. Qi, L., and Sun, D., *Improving the Convergence of Noninterior Point Algorithms for Nonlinear Complementarity Problems*, Mathematics of Computation, Vol. 69, pp. 283–304, 2000.
13. Qi, L., Sun, D., and Zhou, G., *A New Look at Smoothing Newton Methods for Nonlinear Complementarity Problems and Box-Constrained Variational Inequalities*, Mathematical Programming, Vol. 87, pp. 1–35, 2000.
14. Tseng, P., *Analysis of a Noninterior Continuation Method Based on Chen–Mangasarian Smoothing Functions for Complementarity Problems*, Reformulation: Nonsmooth, Piecewise Smooth, Semismooth, and Smoothing Methods, Edited

by M. Fukushima and L. Qi, Kluwer Academic Publishers, Dordrecht, Holland, pp. 381–404, 1997.

15. Xu, S., *The Global Linear Convergence of an Infeasible Noninterior Path-Following Algorithm for Complementarity Problems with Uniform P-Functions*, Mathematical Programming, Vol. 87, pp. 501–517, 2000.
16. Xu, S., *The Global Linear Convergence and Complexity of a Noninterior Path-Following Algorithm for Monotone LCP Based on the Chen–Harker–Kanzow–Smale Smoothing Function*, Technical Report, Department of Mathematics, University of Washington, Seattle, Washington, 1997.
17. Xu, S., and Burke, J. V., *A Polynomial-Time Interior-Point Path-Following Algorithm for LCP Based on Chen–Harker–Kanzow Smoothing Techniques*, Mathematical Programming, Vol. 86, pp. 91–103, 1999.
18. Hotta, K., Inaba, M., and Yoshise, A., *A Complexity Analysis of a Smoothing Method Using CHKS-Functions for Monotone Linear Complementarity Problems*, Computational Optimization and Applications, Vol. 17, pp. 183–201, 2000.
19. Smale, S., *Algorithms for Solving Equations*, Proceedings of the International Congress of Mathematicians, American Mathematical Society, Providence, Rhode Island, pp. 172–195, 1987.
20. Chen, C., and Mangasarian, O. L., *A Class of Smoothing Functions for Nonlinear and Mixed Complementarity Problems*, Computational Optimization and Applications, Vol. 5, pp. 1168–1190, 1996.
21. Luo, X. D., and Tseng, P., *On a Global Projection-Type Error Bound for the Linear Complementarity Problem*, Linear Algebra and Its Applications, Vol. 253, pp. 251–278, 1997.
22. Mangasarian, O. L., and Ren, J., *New Improved Error Bounds for the Linear Complementarity Problem*, Mathematical Programming, Vol. 66, pp. 241–255, 1994.
23. Mathias, R., and Pang, J. S., *Error Bounds for the Linear Complementarity Problem with a P-Matrix*, Linear Algebra and Its Applications, Vol. 132, pp. 123–136, 1990.
24. Tseng, P., *Growth Behavior of a Class of Merit Functions for the Nonlinear Complementarity Problem*, Journal of Optimization Theory and Applications, Vol. 89, pp. 17–37, 1996.
25. Cottle, R. W., Pang, J. S., and Stone, R. E., *The Linear Complementarity Problem*, Academic Press, New York, NY, 1992.
26. Smith, R. L., *Some Interlacing Properties of the Schur Complement of a Hermitian Matrix*, Linear Algebra and Its Applications, Vol. 177, pp. 137–144, 1992.