Possibility of Finding Integer Valued Function Solutions of a Class of Functional Equations for Functions Defined on the Integers

Contents

We consider functional equations involving mappings of the integers into themselves.

1 Integer Solutions of a Functional Equation

We prove the following theorem.

Theorem 1.1 *For every nonzero integer p there is a function*

$$f : \mathbb{Z} \rightarrow \mathbb{Z} \tag{1.1}$$

where

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \cdots, n, -n, \cdots\} \tag{1.2}$$

such that

$$f(f(n)) = n + p \tag{1.3}$$

if p is even. However, if p is an odd integer no solution can exist.

Proof of Theorem. We prove several propositions and a lemma to complete the proof of the theorem.

Proposition 1.1 *If a function f satisfies (1.1) and (1.3), then f is one to one, onto, and has no fixed points.*

Proof. If f were not one to one, then there would be integers n and m with $n \neq m$ and $f(n) = f(m)$. But then we would have the contradiction that both

$$f(f(n)) = f(f(m)) \tag{1.4}$$

and

$$f(f(n)) = n + p \neq m + p = f(f(m)) \tag{1.5}$$

are true.

Next an f satisfying (1.1) and (1.3) has no fixed points. For if there were an n such that $f(n) = n$ we would have

$$n = f(n) = f(f(n)) = n + p \tag{1.6}$$

which contradicts the hypothesis $p \neq 0$.

Next we show that f must be onto. If this were not true there would be an ℓ in $\mathbb{Z}$ such that $f(n) \neq \ell$ for every n in $\mathbb{Z}$. But there is certainly an m in $\mathbb{Z}$ such that

$$\ell = m + p \tag{1.7}$$

and

$$f(f(m)) = m + p = \ell \tag{1.8}$$

Substituting

$$n = f(m) \tag{1.9}$$

into (1.8) giving us the contradition

$$f(n) = \ell \tag{1.10}$$

of our hypothesis.

We prove the following.

Proposition 1.2 *If $p = 2m$ then then the function defined by*

$$f(n) = n + m \tag{1.11}$$

satisfies

$$f(f(n)) = f(n + m) = (n + m) + m = n + (m + m) = n + p \tag{1.12}$$

for all n in $\mathbb{Z}$.

Proof of Proposition. Equation (1.3) is a consequence of the hypothesis (1.3).

Now establish a Lemma which will show that f satisfying (1.3) must map the set of classes of integers which have the same remainder after division by p. These classes are disjoint so that the relation is an equivalence relation.

Lemma 1.1 *For every*

$$j \in \{0, \cdots, \mid p \mid -1\} \tag{1.13}$$

there is an $a_j \in \mathbb{Z}$ with

$$f(a_j) = j \tag{1.14}$$

then (1.3) implies that

$$f(j) = f(f(a_j) = a_j + p \tag{1.15}$$

From (1.15) and (1.3). For every positive integer k we have

$$f(j + (k - 1)p) = a_j + kp \tag{1.16}$$

so that

$$f(j+(k-1)p)\ MODULO\ \mid p \mid = a_j \tag{1.17}$$

and

$$f(a_j+kp) = j+kp \tag{1.18}$$

so that

$$f(a_j+kp)\ MODULO\ \mid p \mid = j \tag{1.19}$$

Proof of Lemma. Let $\mathcal{P}(k)$ be the assertion that (1.16) is valid for the positive integer k and let $\mathcal{Q}(k)$ be the assertion that (1.18) is valid for the positive integer k. By (1.15) which is a consequence of the hypothesis (1.3), the assertion $\mathcal{P}(1)$ is true. By (1.15) which is also a consequence of the definition (1.14) of a_j and the hypothesis (1.3), the assertion $\mathcal{Q}(1)$ is true. Now using the inductive hypothesis suppose the assertions $\mathcal{P}(k)$ and $\mathcal{Q}(k)$ are both true.

By $\mathcal{Q}(k)$ we suppose that (1.18) is true. Applying f to each side of (1.18) and using (1.3) gives

$$f(j+kp) = a_j+(k+1)p \tag{1.20}$$

which is exactly the assertion $\mathcal{P}(k+1)$.

The assertion $\mathcal{P}(k)$ is that (1.16) is valid for the positive integer k. Applying the function f to each side of (1.16) gives us

$$f(f(j+(k-1)p)) = f(a_j+kp) \tag{1.21}$$

Applying (1.3) to (1.21) gives

$$f(a_j+kp) = j+kp \tag{1.22}$$

which is exactly the assertion $\mathcal{Q}(k+1)$. The lemma is proven.

In this lemma every k may be replaced by $-k$. This means that the following is true.

Corollary 1.1 *If a mapping f from $\mathbb{Z}$ into itself satisfies (1.3), then (1.16) and (1.18) hold for every k in $\mathbb{Z}$.*

Proof of Corollary. To start out just observe that there is some integer m in $\mathbb{Z}$ such that

$$f(a_j-p) = m \tag{1.23}$$

So by (1.3)

$$a_j = f(f(a_j-p) = f(m) \tag{1.24}$$

Using the definition of a_j given by (1.14) tells us that

$$j = f(a_j)) = f(f(m)) = m+p \tag{1.25}$$

so that $m = j-p$ and (1.23) yields

$$f(a_j-p) = j-p \tag{1.26}$$

The proof of the corollary proceeds systematically along the lines of the proof of the Lemma.

Now we show how the function f satisfying (1.3) must operate on a family of disjoint equivalence classes. For every integer j in $\mathbb{Z}$ we define

$$\mathcal{C}(j) = \{m \in \mathbb{Z} : m \ MODULO \ \mid p \mid \ = \ j \ MODULO \ \mid p \mid\} \tag{1.27}$$

The definition (1.27) and (1.16) tells us that

$$f : \mathcal{C}(j) \to \mathcal{C}(a_j) \tag{1.28}$$

By the corollary these mappings of f restricted the the equivalence classes are bijections or one to one maps of one equivalence class onto the other. Either these classes are the same or they are different. Suppose they are different. The equation (1.18) and the definition (1.16) tells us that

$$f : \mathcal{C}(a_j) \to \mathcal{C}(j) \tag{1.29}$$

If p is an odd integer, then there are an odd number of pairwise disjoint equivalence classes since there is an odd number of remainders when you divide an integer by an odd number. We can consider f to be a mapping of these equivalence classes onto themselves.

The mapping f on the integers is a composition of pairwise disjoint transpositions since when regarded as a function on the set of equivalence classes is by the corollary its own inverse. Thus, if p is odd, then there must be at least one j where

$$j \in \{0, \cdots, \mid p \mid\} \tag{1.30}$$

such that

$$a_j = j + mp \tag{1.31}$$

Thus, by (1.31), the definition of a_j given by (1.14), and (1.3) we conclude that

$$f(a_j) = j = f(j + mp) = a_j + (m + 1)p = j + mp + (m + 1)p \tag{1.32}$$

This would mean that

$$j = j + (2m + 1)P \tag{1.33}$$

which in turn tells us that

$$(2m + 1)p = 0 \tag{1.34}$$

This means

$$m = -1/2 \tag{1.35}$$

Then equations (1.35) and (1.31) would imply that

$$a_j = j - p/2 \tag{1.36}$$

is not an integer which is a contradiction. If $\mid p \mid$ is odd there is no solution in integers.

References

[1] Banerji, R. and M. D. Mesarovic (Editors) *Theoretical Approaches to Nonnumerical Problem Solving* New York: Springer Verlag (1970) New York: McGraw Hill (1964)