# Challenge of the Week

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## Week 3

You put a penny on a square table. After measuring the distances, you find that it is 13 inches from one corner, 25 inches from the opposite corner, and 19 inches from another corner. What is the area of the top of the table?

## Solution

Label the vertices of the square $A$, $B$, $C$, and $D$ where $P$ is the interior point such that $AP = 13$, $BP = 19$ and $CP = 25$. Let $P'$ be the image of point $P$ after it is rotated $90^\circ$ clockwise about point $B$. Then $\angle PBP' = 90^\circ$ and $PB = BP' = 19$ and thus triangle $\triangle PBP'$ is an isosceles right triangle with $\angle PP'B = 45^\circ$ and $PP' = 19\sqrt{2}$. By rotational symmetry $CP' = AP = 13$. Let $\theta = \angle CPP'$. Then by the law of cosines we have:

$$13^2 = 25^2 + 2 \cdot 19^2 - 2 \cdot 13 \cdot 19\sqrt{2}\cos\theta$$

From which we obtain

$$\cos\theta = \frac{31\sqrt{2}}{50}$$

and furthermore

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{961}{1250}} = \frac{17\sqrt{2}}{50}$$

Applying the law of cosines to triangle $\triangle BPC$ we see

$$BC^2 = 19^2 + 25^2 - 2 \cdot 19 \cdot 25\cos(\theta + 45^\circ) = 986 - 950(\cos\theta\cos 45^\circ - \sin\theta\sin 45^\circ) =$$
$$986 - 950\left(\frac{31\sqrt{2}}{50} \cdot \frac{\sqrt{2}}{2} - \frac{17\sqrt{2}}{50} \cdot \frac{\sqrt{2}}{2}\right) = 986 - 950 \cdot \frac{14}{50} = 720$$

Notice that $BC^2$ is precisely the area of the square top which as found above is 720 $\text{in}^2$.