Challenge of the Week November 27–December 3 2007

Problem

Consider the real polynomial $p(x) = 1 + ax + bx^2 + ax^3 + x^4$. Find the coefficients a and b that minimize the quantity $a^2 + b^2$ such that $p(x)$ has at least one real root.

Hint: Since $p(x)$ is palindromic it is possible to deal with a polynomial with half the degree by considering the quantity $p(x)/x^2$ and making the substitution $y = x + 1/x$.

Solution

(Thanks to Dustin)

If $p(x)$ has a real root, then so does $p(x)/x^2$, since we know $x = 0$ isn't a root. From the hint, we can make the substitution $y = x + 1/x$. (To make the substitution: if $y = x + 1/x$, then $y^2 = x^2 + 2 + 1/x^2$, and we see $p(x)/x^2 = y^2 + ay + b - 2$.)

However, we have to be careful. This substitution is invertible if and only if $|y| \geq 2$. So $p(x)$ will have a real root if and only if $y^2 + ay + (b - 2)$ has a real root c with $|c| \geq 2$. Using the quadratic formula, we know this quadratic has a real root exactly when the discriminant $a^2 - 4b + 8 \geq 0$; we also require that $2\,|y| = \left|-a \pm \sqrt{a^2 - 4b + 8}\right| \geq 4$.

We are seeking to minimize $a^2 + b^2$ subject to these constraints. The minimum will occur on the boundary (the unconstrained minimum occurs when $a = b = 0$ which is not feasible), so we can let $-a \pm \sqrt{a^2 - 4b + 8} = \pm 4$. Simplifying this leads to $b = \pm 2a - 2$. Assuming this, then if we plug it into $a^2 - 4b + 8 \geq 0$, we get the inequality $a^2 - 4(\pm 2a - 2) + 8 \geq 0$, which simplifies down to $(a \pm 4)^2 \geq 0$, which always holds.

So we seek to minimize $a^2 + b^2$, subject to $b = \pm 2a - 2$. Plugging this in for b, we are seeking to minimize $a^2 + (\pm 2a - 2)^2 = 5a^2 \pm 8a + 4$. This has a minimum value when $a = \pm 4/5$. From here, we get $b = -2/5$ and a minimum of $a^2 + b^2 = 4/5$. You can check that when a and b are these values, then our original quartic has a root (namely $x = 1$, and $x = -1$).

So to state the answer plainly, the minimum value of $a^2 + b^2$ is $4/5$.

Solvers

Aaron Dilley (undergraduate), Dustin Moody, Gary Raymond (graduate).