

Problem: (12/4/2007—12/10/2007)

Pick m points on the positive x -axis and n points on the positive y -axis. Draw a line segment from each of the points on the x -axis to each of the points on the y -axis. Assume that no three lines meet at a point except at endpoints. Excluding the endpoints, how many points of intersection are there?

Solution 1:

Label the points on the x -axis $x_1 < x_2 < \dots < x_m$, and the points on the y -axis $y_1 < y_2 < \dots < y_n$. Now start drawing the line segments one at a time, counting the intersections along the way.

- When we join y_1 to each of the x_i , we get 0 intersections.
- Next join y_2 to the x_i 's. We get

$$(m-1) + (m-2) + \dots + 2 + 1 = \frac{m(m-1)}{2}$$

intersections.

- Next join y_3 to the x_i 's. We get

$$2(m-1) + 2(m-2) + \dots + 2(2) + 2(1) = 2 \frac{m(m-1)}{2}$$

intersections.

- Continue this way until we join y_n to each of the x_i 's. Here there are

$$(n-1)(m-1) + (n-1)(m-2) + \dots + (n-1)(2) + (n-1)(1) = (n-1) \frac{m(m-1)}{2}$$

intersections.

Adding all the intersections up, we get

$$(1 + 2 + \dots + (n-1)) \frac{m(m-1)}{2} = \frac{n(n-1)}{2} \frac{m(m-1)}{2}$$

intersections total.

Solution 2:

Again label the points on the x-axis $x_1 < x_2 < \dots < x_m$, and the points on the y-axis $y_1 < y_2 < \dots < y_n$.

Observe that there is a bijection (a one-to-one correspondence) between points of intersection and two unordered pairs of endpoints $(x_i, x_j), (y_k, y_l)$. To see this bijection: given a point of intersection, follow the two lines that meet at that point back to their endpoints; this yields two pairs of endpoints. Conversely, given two unordered pairs of endpoints, reorder the points so that $x_i < x_j$ and $y_k < y_l$. Then the two segments $x_i—y_l$ and $x_j—y_k$ intersect.

So, counting the number of intersections is equivalent to counting the number of ways to pick two unordered pairs of endpoints. There are $\binom{m}{2}$ ways to pick the two x 's, and $\binom{n}{2}$ ways to pick the y 's. Combining all possible choices for x 's and y 's gives

$$\binom{m}{2} \binom{n}{2} = \frac{n(n-1)}{2} \frac{m(m-1)}{2} \text{ intersections.}$$

(The notation $\binom{a}{b}$ denotes the number of ways to choose b things from a total things. It may be computed by $\binom{a}{b} = \frac{a!}{b!(a-b)!}$).

Correct solutions:

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