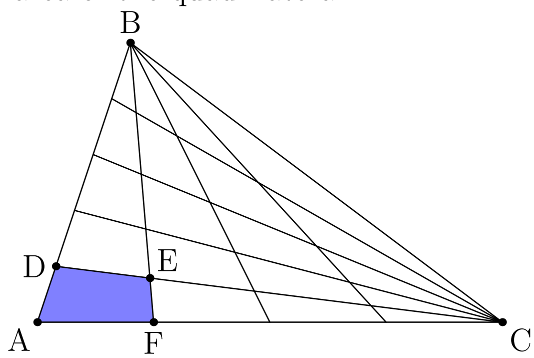
Challenge of the Week

January 8–January 14, 2008

Problem

Suppose ABC is a general triangle with area 1. Divide side AC into 4 equal parts and connect B (with straight lines) to each quarter of AC. Similarly, divide side AB into 5 equal parts and connect each fifth back to point C. Let D be the point that is 1/5 of the way from A to B, and let F be the point 1/4 of the way from A to C, and let E be the intersection of CD and BF. What is the area of the quadrilateral $ADEF$?



Solution

The area is 31/380.

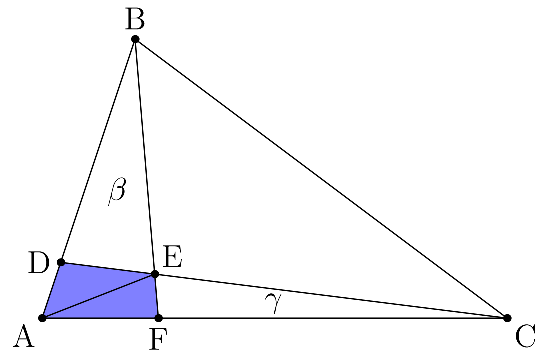
Here's an elegant proof by Alan Jamison.

$\triangle ADC$ has base AD and some height h. Treating AB as the base for $\triangle ABC$, we see that $\triangle ABC$ has the same height h as $\triangle ABC$, but (by hypothesis) a base five times as long. Thus

$$\text{area}(\triangle ADC) = \frac{1}{5}\text{area}(\triangle ABC) = \frac{1}{5}.$$

By a similar argument,

$$\text{area}(\triangle AFB) = \frac{1}{4}.$$



Let α denote the area of $ADEF$, β the area of $\triangle BDE$, and γ the area of $\triangle CFE$. Clearly $\alpha + \beta = \frac{1}{4}$ and $\alpha + \gamma = \frac{1}{5}$, or equivalently,

$$\beta = \frac{1}{4} - \alpha \qquad \text{and} \qquad \gamma = \frac{1}{5} - \alpha.$$

Notice that $\triangle BDE$ and $\triangle ADE$ share a common height with $\triangle BDE$'s base four times as long as $\triangle ADE$'s, so

$$\text{area}(\triangle ADE) = \frac{1}{4}\beta = \frac{1}{16} - \frac{\alpha}{4}.$$

By a similar argument, we see that

$$\text{area}(\triangle AFE) = \frac{1}{3}\gamma = \frac{1}{15} - \frac{\alpha}{3}.$$

But, $\alpha = \text{area}(\triangle ADE) + \text{area}(\triangle AFE)$, so we find the equation

$$\alpha = \frac{1}{16} - \frac{\alpha}{4} + \frac{1}{15} - \frac{\alpha}{3}.$$

Solving this, we obtain $\alpha = \dfrac{31}{380}$.

If AC and AB are subdivided into n and m equal pieces, respectively, then the same arguments lead to

$$\alpha = \frac{1}{n(m-1)} + \frac{\alpha}{m(n-1)} - \alpha\left(\frac{1}{m-1} + \frac{1}{n-1}\right)$$

giving the solution

$$\alpha = \frac{2nm - (m+n)}{nm(nm-1)}.$$