Challenge of the Week

January 15–January 21 2008

Problem:

A function f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$. Furthermore, $f(a) = a$ and $f(b) = b$. Show that there are two points x_1, x_2 such that

$$a < x_1 < x_2 < b \qquad \text{and} \qquad \frac{1}{f'(x1)} + \frac{1}{f'(x2)} = 2.$$

Solution:

First find a point c where the function is halfway between a and b. By the Intermediate Value theorem, there exists c such that

$$a < c < b \qquad \text{and} \qquad f(c) = (a+b)/2.$$

The average slope of f from a to c is

$$m_1 = \frac{f(c) - f(a)}{c - a} = \frac{(a+b)/2 - a}{c - a} = \frac{b - a}{2(c - a)}.$$

The average slope of f from c to b is

$$m_2 = \frac{f(b) - f(c)}{b - c} = \frac{b - (a+b)/2}{b - c} = \frac{b - a}{2(b - c)}.$$

These slopes are special: they satisfy the property

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{2(c - a) + 2(b - c)}{b - a} = 2.$$

This is exactly the property that we want for the slopes at the two unknown points x_1 and x_2. How do we know such points exist? Since the function is differentiable and has average slope m_1 from a to c, the Mean Value theorem guarantees there's an x_1 so that $a < x_1 < c$ and $f'(x_1) = m_1$. Similarly it guarantees an x_2 with $c < x_2 < b$ and $f'(x_2) = m_2$. Done.