Challenge Of the Week

January 22—January 28, 2008

Problem:

There are 10000 people in a stadium, and a (very large) number is shown to everyone. The first person says the number is divisible by 1; the second says the number is divisible by 2; a third says the number is divisible by 3. This continues until the last person says the number is divisible by 10000. If exactly two consecutive people lied, who were they?

Solution:

Let's call the huge unknown number n, and call the two liars k and $k+1$.

Suppose $k \leq 5000$. This means that n is not divisible by k. But then n is also not divisible by $2k$, so person $2k$ would be lying as well. Since only two people are lying, this is impossible. Thus $k > 5000$.

Now suppose that k factors as $k = rs$, with r and s relatively prime. Then since $k \nmid n$, either $r \nmid n$ or $s \nmid n$. This is impossible, since either r or s would be a liar before 5000. We must conclude that k is a prime power. Applying the same argument to $k+1$, we know $k+1$ is also a prime power.

Since one of k and $k+1$ is even, one of the liars must be a power of 2. The only power of two between 5000 and 10000 is 8192. Thus one liar is 8192 and the other is 8191 or 8193. Now, $8193 = 3 \times 2371$ is not a prime power as 2371 is not divisible by 3, so the other liar must be 8191 (a prime number).