Challenge Of the Week

January 29—February 4, 2008

Problem:

Prove that any set of three distinct integers includes two integers x and y such that $f(x, y) = x^3y - xy^3$ is divisible by 10.

Solution:

Let the distinct integers by x, y, and z.

One approach is to use brute force. Working modulo 10, we can check that for every possible residue class for x, y, and z works. That is, we can reduce the problem to the finite list of cases where $0 \le x, y, z \le 9$. Checking all 1000 possibilies shows that no matter what, one of $f(x, y), f(x, z)$, or $f(y, z)$ is divisible by 10.

We can also reason this out by hand.

Note for any integers x, y that $f(x, y) = xy(x + y)(x - y)$ is congruent to zero mod 2 since one of the three of $x, y, x + y$ must be even. Hence we need only show that given integers x, y, z, one of $f(x, y), f(x, z), f(y, z)$ is congruent to zero mod 5. In other words, at least one of the factors $x, y, z, x + y, x + z, y + z, x - y, x - z, y - z$ is congruent to zero mod 5.

If 5 divides x, y, or z, then we're done. So assume x, y, and z are not congruent to 0 mod 5. If two of them are in the same residue class mod 5, say for example $x = y$ mod 5, then the difference $x - y = 0$ mod 5, and we're done. Finally, assume that x, y, and z are different mod 5 and not congruent to 0. They must lie in the set $\{1, 2, 3, 4\}$, and we have to choose three different ones. There are only four ways to do this: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$. In all cases, we have two numbers whose sum is 0 mod 5, so one of $x+y, x+z, y+z$ is congruent to 0 mod 5.