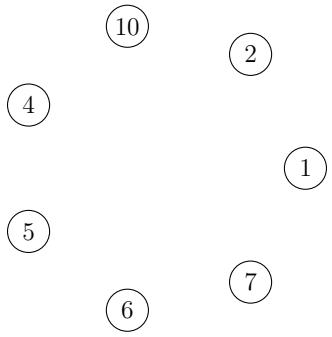
Challenge Of the Week

February 5—February 11, 2008

Problem:

The seven integers in the circular arrangement of seven disks shown below have the property taht every one of the integers $1, 2, \ldots, 14$ is either a disk or else is the sum of the integers in two adjacent disks. Can you replace the numbers in the disks with integers, none of which is 5, and retain the same property?



Solution:

We can find a solution either with a computer or by trial and error, but there are a few observations we can make to narrow down our search. Let the seven numbers be $a_1, a_2, \ldots, a_7$.

First note that as there are only seven numbers and seven pairs of adjacent numbers, and we want to get 14 different values, the a_i's must be distinct. In fact, all the a_i's must be between 1 and 14, and all the sums must be between 1 and 14.

From this, one of the a_i's is 1, and another is 2, since there's no other way to get these numbers as sums. Next, we can't have any of the a_i's be 13 or 14. To see this, if $a_i \geq 13$, then the smallest possible adjacent numbers are 1 and 2, which would mean that one of the sums is at least 15, which is impossible.

If we add the the numbers and the pairs of adjacent numbers, then we can get a restriction

on the sum of the a_i's, as well:

$$(a_1 + \cdots + a_7) + ((a_1 + a_2) + \cdots (a_6 + a_7) + (a_7 + a_1)) = 1 + 2 + \cdots + 14$$

$$3\sum_{i=1}^{7} a_i = 105$$

$$\sum_{i=1}^{7} a_i = 35$$

Finally, we can consider the symmetry of the problem. We can rotate or reflect a solution to get another solution. So we can fix the location for disk "1", and assume (by reflecting) that disk "2" is in one of three positions.

From here, we may easily find a solution via trial and error. Here are all the solutions, starting at disk 1, and reading around (either clockwise or counterclockwise).

(with no 5's)	$1, 12, 2, 3, 6, 4, 7$
	$1, 12, 2, 6, 3, 7, 4$
	$1, 11, 2, 3, 4, 6, 8$
	$1, 11, 2, 8, 6, 3, 4$
	$1, 6, 2, 3, 9, 4, 10$
	$1, 6, 2, 9, 3, 10, 4$
	$1, 6, 3, 11, 2, 8, 4$
(containing 5's)	$1, 10, 2, 5, 8, 6, 3$
	$1, 9, 4, 2, 5, 3, 11$
	$1, 2, 6, 7, 5, 4, 10$
	$1, 2, 10, 4, 5, 6, 7$
	$1, 2, 5, 6, 8, 4, 9$
	$1, 2, 8, 4, 9, 5, 6$
	$1, 2, 9, 5, 8, 4, 6$
	$1, 3, 10, 2, 5, 6, 8$
	$1, 12, 2, 8, 3, 4, 5$
	$1, 11, 2, 4, 3, 5, 9$