Challenge Of the Week

February 12—February 18, 2008

Problem

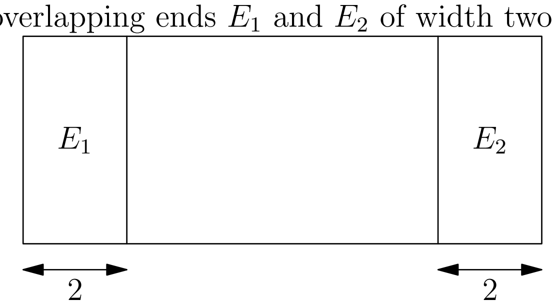
Given a $(2m+1) \times (2n+1)$ checkerboard in which the four corners are black squares, show that if one removes any one red square and any two black squares, the remaining board is coverable with dominoes (that is, by 1 x 2 rectangles).

Solution

We'll call a $(2m+1) \times (2n+1)$ checkerboard with one red square and two black squared removed a *deleted checkerboard*.

First note we can easily verify that all 3×3 deleted checkerboards can be tiled. (After accounting for symmetry, there are only a few cases to check.)

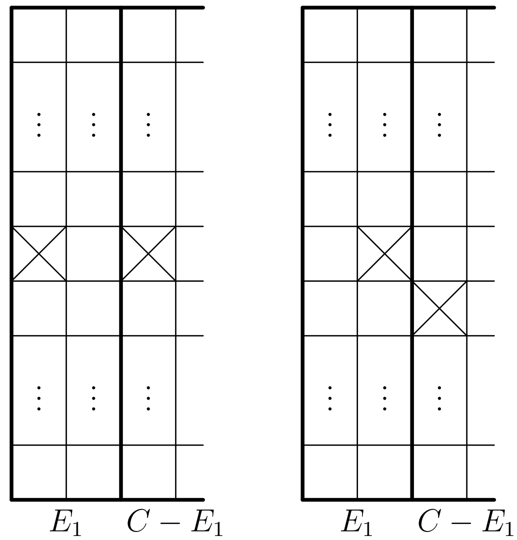
Now proceed by induction. Given a $(2m+1) \times (2n+1)$ deleted checkerboard C, we may assume that any smaller $(2k+1) \times (2l+1)$ deleted checkerboard contained in C may be covered in dominoes. Since one of the dimensions of C is of length at least five, C has two opposeitely placed, non-overlapping ends E_1 and E_2 of width two.



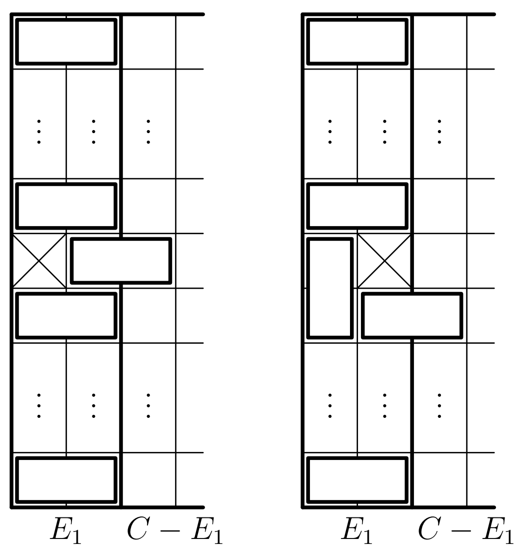
As there are only three deleted squares, one of the ends must have 0 or 1 deleted squares. Without loss of generality, call this end E_1.

If E_1 contains no deleted squares, then $C - E_1$ contains all three. By induction, we can cover $C - E_1$ with dominoes, and then cover E_1 with horizontal dominoes to make a covering for C.

Otherwise, suppose E_1 contains exactly one deleted square. In this case, pick the closest square in $C - E_1$ of the same color, and delete it, as shown below:



Now by induction, there is a domino covering for $C - E_1$ with this deletion. To recover a covering for the original problem, use the tiling just found, and tile E_1 and the extra deleted square as shown below.



This procedure will only fail in the case where the extra square in $C-E_1$ was already deleted. If there was a red square missing in E_1, then this problem will not arise. If there was a black square missing in E_1, and a problematic nearest square in $C - E_1$, then there is at most one square missing in E_2 (and no problematic near squares) so we can apply the above argument to E_2 rather than to E_1 to get a covering.