Problem:

(a) Show that if $n$ is a triangular number, then so is $9n + 1$. (Triangular numbers are $1, 3, 6, 10, \ldots, k(k + 1)/2, \ldots$)

(b) Find other numbers $a$ and $b$ so that $an + b$ is triangular whenever $n$ is.

Solution:

A number $n$ is triangular if and only if $n = k(k + 1)/2$ (for some $k$). Observe that if $n$ is triangular, then

$$9n + 1 = \frac{9k(k + 1)}{2} + 1 = \frac{(3k + 1)((3k + 1) + 1)}{2}$$

is a triangular number as well. A pretty alternative is the following “proof without words”:

![Proof without words](image)

For the second part, suppose that $an + b$ is triangular whenever $n$ is triangular. That is, if $n = k(k + 1)/2$ and $an + b = j(j + 1)/2$, we have

$$a\frac{k(k + 1)}{2} + b = \frac{j(j + 1)}{2}.$$ 

An easy way to find some solutions is to realize that this is quadratic in both $k$ and $j$, so it is reasonable to assume that $j$ is a linear function of $k$. Substituting in $j = ck + d$, and
matching up coefficients, we can solve for \(a\) and \(b\):

\[
\begin{align*}
(k^2) & \quad a = c^2 \\
(k) & \quad a = c(2d + 1) \\
(1) & \quad b = d(d + 1)/2, \text{ for } d = 0, 1, 2, \ldots
\end{align*}
\]

Solving these equations finds two families of solutions; when \(c = 0\) we get

\[
a = 0 \quad b = \frac{d(d + 1)}{2}, \quad \text{where } d = 0, 1, 2, \ldots;
\]

and when \(c \neq 0\), we get

\[
a = (2d + 1)^2, \quad b = \frac{d(d + 1)}{2}, \quad \text{where } d = 0, 1, 2, \ldots.
\]

This approach also generalizes nicely to finding triangular numbers of the form \(a_2n^2 + a_1n + a_0\) (and higher degrees, as well).

Now for a subtle detail. How do we know we’ve found all the solutions? The construction above relied on the assumption that \(j\) is a linear function of \(k\). If we can show that this must be the case, then we know there are no other solutions.

Going back to the original condition,

\[
a\frac{k(k + 1)}{2} + b = \frac{j(j + 1)}{2},
\]

we can explicitly solve for \(j\) using the quadratic formula:

\[
j = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4ak^2 + 4ak + 8b}.
\]

Now, for \(j\) to be an integer, the expression under the radical must be square for any integer \(k\). Let \(f(k) = 1 + 4ak^2 + 4ak + 8b\). We can factor this as \(f(k) = 4a(k - z_a)(k - z_b)\), with \(z_a, z_b\) possibly complex. But \(f(x + 1) - f(k) = 8ak + 4a - 4a(z_a - z_b)\) must be an integer, so \(4a(z_a + z_b)\) is an integer. Since \(f(z_a) = 0, f(z_a + 1) = 4a(1 - z_b)\) is also an integer. Hence \(z_b = \frac{m_b}{4a}\) for \(m_b \in \mathbb{Z}\) and similarly \(z_a = \frac{m_a}{4a}\) for \(m_a \in \mathbb{Z}\). But the product of the roots is \(8b + 1\), an integer, so \(z_a\) and \(z_b\) are integers.

If \(z_a \neq z_b\) then there is some \(k\) so that \(f(k)\) is nonsquare. For example, suppose without loss of generality that \(z_a > z_b\) and let \(k = p + z_a\), where \(p\) is a prime larger than either \(4a\) or \(z_a - z_b\). Then \(f(k) = 4ap(p + z_a - z_b)\) has only one factor of \(p\).

So we conclude that \(f(k) = 4a(k-z)^2\), for some integer \(z = z_a = z_b\). then \(j = -\frac{1}{2} \pm \sqrt{a(k-z)}\). Regardless whether we choose the + or the − for the square root, \(j\) is a linear function of \(k\).