Challenge Of the Week

February 19—February 25, 2008

Problem:

(a) Show that if n is a triangular number, then so is $9n + 1$. (Triangular numbers are $1, 3, 6, 10, \ldots, k(k + 1)/2, \ldots$)

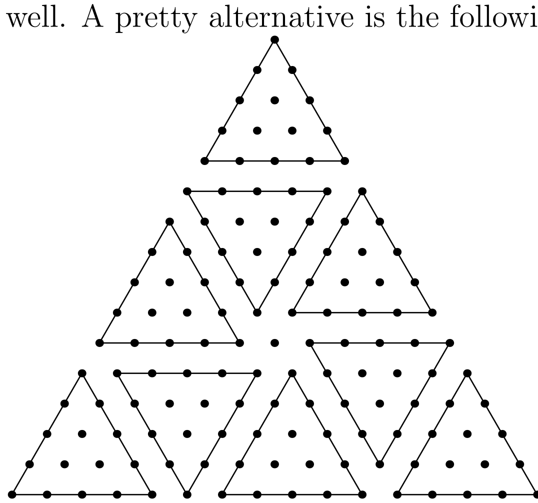
(b) Find other numbers a and b so that $an + b$ is triangular whenever n is.

Solution:

A number n is triangular if and only if $n = k(k + 1)/2$ (for some k). Observe that if n is triangular, then

$$9n + 1 = \frac{9k(k + 1)}{2} + 1 = \frac{(3k + 1)((3k + 1) + 1)}{2}$$

is a triangular number as well. A pretty alternative is the following "proof without words":



For the second part, suppose that $an + b$ is triangular whenever n is triangular. That is, if $n = k(k + 1)/2$ and $an + b = j(j + 1)/2$, we have

$$a\frac{k(k + 1)}{2} + b = \frac{j(j + 1)}{2}.$$

an easy way to find *some* solutions is to realize that this is quadratic in both k and j, so it is reasonable to assume that j is a linear function of k. Substituting in $j = ck + d$, and matching up coefficients, we can solve for a and b:

$$\begin{array}{ll} (k^2) & a = c^2 \\ (k) & a = c(2d+1) \\ (1) & b = d(d+1)/2, \quad \text{for } d = 0,1,2,\ldots \end{array}$$

Solving these equations finds two families of solutions; when $c = 0$ we get

$$a = 0 \qquad b = \frac{d(d+1)}{2}, \qquad \text{where } d = 0,1,2,\ldots;$$

and when $c \neq 0$, we get

$$a = (2d+1)^2, \qquad b = \frac{d(d+1)}{2}, \qquad \text{where } d = 0,1,2,\ldots.$$

This approach also generalizes nicely to finding triangular numbers of the form $a_2n^2+a_1n+a_0$ (and higher degrees, as well).

Now for a subtle detail. How do we know we've found *all* the solutions? The construction above relied on the assumption that j is a linear function of k. If we can show that this must be the case, then we know there are no other solutions.

Going back to the original condition,

$$a\frac{k(k+1)}{2} + b = \frac{j(j+1)}{2},$$

we can explicitly solve for j using the quadratic formula:

$$j = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4ak^2 + 4ak + 8b}.$$

Now, for j to be an integer, the expression under the radical must be square for any integer k. Let $f(k) = 1 + 4ak^2 + 4ak + 8b$. We can factor this as $f(k) = 4a(k - z_a)(k - z_b)$, with z_a, z_b possibly complex. But $f(x+1) - f(k) = 8ak + 4a - 4a(z_a - z_b)$ must be an integer, so $4a(z_a + z_b)$ is an integer. Since $f(z_a) = 0$, $f(z_a + 1) = 4a(1 - z_b)$ is also an integer. Hence $z_b = \frac{m_b}{4a}$ for $m_b \in \mathbb{Z}$ and similarly $z_a = \frac{m_a}{4a}$ for $m_a \in \mathbb{Z}$. But the product of the roots is $8b+1$, an integer, so z_a and z_b are integers.

If $z_a \neq z_b$ then there is some k so that $f(k)$ is nonsquare. For example, suppose without loss of generality that $z_a > z_b$ and let $k = p + z_a$, where p is a prime larger than either $4a$ or $z_a - z_b$. Then $f(k) = 4ap(p + z_a - z_b)$ has only one factor of p.

So we conclude that $f(k) = 4a(k-z)^2$, for some integer $z = z_a = z_b$. then $j = -\frac{1}{2} \pm \sqrt{a}(k-z)$. Regardless whether we choose the $+$ or the $-$ for the square root, j is a linear function of k.