Challenge Of the Week

February 26—March 4, 2008

Problem:

Suppose there are three dice with 18 distinct integers on the faces of the three cubes. They are numbered in such a way so that it is possible to obtain any of the integers between 1 and 216, inclusive, as the sum of the integers on the top faces of these three dice with a single throw.

What can be the minimum value for the largest of these integers? (Hint: integers can be negative!)

Solutions:

Most people found best (smallest) maximum to be 75, assuming that the numbers on the dice came from arithmetic progressions. This is the sort of argument I expected. However, the best value is 74. I'll give a short proof of this, and then the original solution by Dustin that led to the value of 74.

Arithmetic Progressions Only

Here's a sketch of the argument I expected. I'm not filling in the details, since we can do better than 75. Suppose the dice are labeled:

Die 1:	$x, x+a, x+2a, \ldots, x+5a$
Die 2:	$y, y+b, y+2b, \ldots, y+5b$
Die 3:	$z, z+c, z+2c, \ldots, z+5c$

then it's not hard to see that we must have $a = 1$, $b = 6$ and $c = 36$ to get 216 consecutive numbers. One labeling of the dice that works is

Die 1:	1, 2, 3, 4, 5, 6
Die 2:	0, 6, 12, 18, 24, 30
Die 3:	0, 36, 72, 108, 144, 180

From here, we can get other solutions by adding any number k to every face on one die, and simultaneously subtracting k to every face on another die. With some argument, the following solution may be found, and may be shown optimal:

Die 1: 69, 70, 71, 72, 73, 74
Die 2: 37, 43, 49, 55, 61, 67
Die 3: −105, −69, −33, 3, 39, **75**

General Solution

The smallest possible value for the largest number on the dice is 74.

As a first step, we show that the largest value is at least 74. Suppose by way of contradiction that all the sides on the dice have values 73 or less (and are distinct).

The only way we can roll 216 is using the largest possible numbers 71 + 72 + 73. Thus these must be the largest sides on each of the the dice. The dice must be:

$$(\ldots, 71), \qquad (\ldots, 72), \qquad (\ldots, 73)$$

The only way to roll 215 is using both the numbers 72 and 73 on the second and third dice. If we don't use these numbers, then the next largest number available to be on the dice is 70, but then the highest we could roll is 71 + 70 + 73 = 214, which is too small. Since we use 72 and 73, the first die must have a 70. Thus the dice must look like this:

$$(\ldots, 70, 71), \qquad (\ldots, 72), \qquad (\ldots, 73)$$

By the same reasoning, the only way to roll a 214 is using 72 and 73 again. If we don't use these numbers, then the largest roll we can make is only 71 + 69 + 73 = 213, which is too small. Using a 72 and 73, the first die must have a 69. Thus the dice must look like this:

$$(\ldots, 69, 70, 71), \qquad (\ldots, 72), \qquad (\ldots, 73)$$

Reasoning the same way as above, the first die must have a 68 to be able to roll a 213, a 67 to roll a 212, and a 66 to roll a 211. Thus the dice must look like this:

$$(66, 67, 68, 69, 70, 71), \qquad (\ldots, 72), \qquad (\ldots, 73)$$

How can we roll a 210? We can't use both 72 and 73, since we used them in every possible combination with the first die to get the numbers 211 through 216. The next largest number we can use on the last two dice is 65. The largest total we can then make is 71 + 65 + 73 = 209, which is too small. In summary: *If the largest number is only 73, then there is no way to roll a 210. Thus the largest number must be 74 or larger.*

The second part of the proof is easy:* check that the numbering

$$(-52, -43, -34, 56, 65, \mathbf{74}), \qquad (9, 12, 15, 63, 66, 69), \qquad (44, 45, 46, 71, 72, 73)$$

gives a solution. This shows that the best largest value is 74 or less.

Now we've bounded the best maximum value to be 74 or less and also 74 or greater. Thus the best maximum value is 74.

*easy, in the sense it's easy to check that this works. It's not so easy to magically find this numbering in the first place!

An algebraic (and computer-aided) solution

How did we get the solution above? This takes some clever thinking and a computer algebra system.

Suppose the numbers on die 1 are $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$, the numbers on die 2 are $b_1 > \cdots > b_6$, and the numbers on die 3 are $c_1 > \cdots > c_6$.

Then we have the following identity:

$$(x^{a_1} + \cdots + x^{a_6})(x^{b_1} + \cdots + x^{b_6})(x^{c_1} + \cdots + x^{c_6}) = (x^1 + x^2 + \cdots + x^{216}).$$

Note we are allowing the exponents to be negative so that the values on the dice can be negative as well.

Note that we can multiply $(x^{a_1}+\cdots+x^{a_6})$ by x^d, and $(x^{b_1}+\cdots+x^{b_6})$ by x^e, and $(x^{c_1}+\cdots+x^{c_6})$ by x^f, as long as $d+e+f=0$.

This allows us to start with all possible factorizations of $x^1 + x^2 + \cdots + x^{216}$ into a product of 3 hexanomials with all the exponents positive, and then by playing with d, e, and f to try to get other possibilities. (Actually, it's computationally and aesthetically nicer to use $1 + x + x^2 + \cdots + x^{215}$ and then we need $d+e+f=1$.)

Using maple, here are all possible factorizations of the desired form:

$$(x^{45}+x^{36}+x^{27}+x^{18}+x^9+1)(x^{60}+x^{57}+x^{54}+x^6+x^3+1)(x^{110}+x^{109}+x^{108}+x^2+x^1+1)$$
$$(x^{45}+x^{36}+x^{27}+x^{18}+x^9+1)(x^{114}+x^{111}+x^{108}+x^6+x^3+1)(x^{56}+x^{55}+x^{54}+x^2+x^1+1)$$
$$(x^{72}+x^{63}+x^{54}+x^{18}+x^9+1)(x^{33}+x^{30}+x^{27}+x^6+x^3+1)(x^{110}+x^{109}+x^{108}+x^2+x^1+1)$$

⋮

(there are 71 total ways)

⋮

$$(x^{148}+x^{76}+x^4+x^{144}+x^{72}+1)(x^{49}+x^{25}+x^1+x^{48}+x^{24}+1)(x^{18}+x^{10}+x^2+x^{16}+x^8+1)$$
$$(x^{148}+x^{76}+x^4+x^{144}+x^{72}+1)(x^{50}+x^{26}+x^2+x^{48}+x^{24}+1)(x^{17}+x^9+x^1+x^{16}+x^8+1)$$

[One way to find all these possibilities is to start with the factorization of

$$\begin{aligned}1+x+\cdots+x^{215} = {} & (x+1)(x^2+1)(x^2-x+1)(x^2+x+1)(x^4+1)\\ & \cdot(x^4-x^2+1)(x^6-x^3+1)(x^6+x^3+1)(x^8-x^4+1)\\ & \cdot(x^{12}-x^6+1)(x^{18}-x^9+1)(x^{18}+x^9+1)\\ & \cdot(x^{24}-x^{12}+1)(x^{36}-x^{18}+1)(x^{72}-x^{36}+1)\end{aligned}$$

And try multiplying combinations of products in all possible ways.]

For each of these 71 possibilities, we can let d, e, and f vary, and check to see when we get the smallest maximum value with distinct exponents. Note, this restricts d, e, and f to a finite (albeit very large) set of values to try, so this guarantees we'll find all solutions. See the list at the end.

To illustrate, for the factorization

$$\begin{aligned} 1 + x + x^2 + \cdots + x^{215} &= (x^{126} + x^{117} + x^{108} + x^{18} + x^9 + 1) \\ &\quad \cdot (x^{60} + x^{57} + x^{54} + x^6 + x^3 + 1) \\ &\quad \cdot (x^{29} + x^{28} + x^{27} + x^2 + x^1 + 1) \end{aligned}$$

we can let $d = -52, e = 9$, and $f = 44$ to get

$$\begin{aligned} x(1 + x + \cdots + x^{215}) &= x^{-52}(x^{126} + x^{117} + x^{108} + x^{18} + x^9 + 1) \\ &\quad \cdot x^9(x^{60} + x^{57} + x^{54} + x^6 + x^3 + 1) \\ &\quad \cdot x^{44}(x^{29} + x^{28} + x^{27} + x^2 + x^1 + 1) \\ x + x^2 + \cdots + x^{216} &= (x^{74} + x^{65} + x^{56} + x^{-34} + x^{-43} + x^{-52}) \\ &\quad \cdot (x^{69} + x^{66} + x^{63} + x^{15} + x^{12} + x^9) \\ &\quad \cdot (x^{73} + x^{72} + x^{71} + x^{46} + x^{45} + x^{44}) \end{aligned}$$

So the dice are numbered like this:

$$(74, 65, 56, -34, -43, -52), \qquad (69, 66, 63, 15, 12, 9), \qquad (73, 72, 71, 46, 45, 44)$$

Note all 18 numbers are distinct, with a maximum of 74. Using a computer to check all the thousands of variations for d, e, and f for each of the 71 different factorizations shows that we can't do better than 74.

All best solutions

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(62, 63, 64, 68, 69, 70)  ( 23, 26, 47, 50, 71, 74)  (-84, -72, -12,  0, 60, 72)
(62, 63, 64, 68, 69, 70)  ( 12, 24, 36, 48, 60, 72)  (-73, -70,  -1,  2, 71, 74)
(62, 63, 64, 68, 69, 70)  ( -1,  2, 35, 38, 71, 74)  (-60, -48, -36, 48, 60, 72)
(60, 61, 64, 65, 68, 69)  ( 23, 25, 47, 49, 71, 73)  (-82, -70, -10,  2, 62, 74)
(60, 61, 64, 65, 68, 69)  ( 14, 26, 38, 50, 62, 74)  (-73, -71,  -1,  1, 71, 73)
(60, 61, 64, 65, 68, 69)  ( -1,  1, 35, 37, 71, 73)  (-58, -46, -34, 50, 62, 74)
(59, 61, 63, 65, 67, 69)  ( 24, 25, 48, 49, 72, 73)  (-82, -70, -10,  2, 62, 74)
(59, 61, 63, 65, 67, 69)  ( 14, 26, 38, 50, 62, 74)  (-72, -71,   0,  1, 72, 73)
(59, 61, 63, 65, 67, 69)  (  0,  1, 36, 37, 72, 73)  (-58, -46, -34, 50, 62, 74)
(56, 57, 62, 63, 68, 69)  ( 34, 36, 38, 70, 72, 74)  (-89, -71, -17,  1, 55, 73)
(56, 57, 62, 63, 68, 69)  ( 16, 18, 20, 70, 72, 74)  (-71, -53, -35, 37, 55, 73)
(56, 57, 62, 63, 68, 69)  (-17,  1, 19, 37, 55, 73)  (-38, -36, -34, 70, 72, 74)
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(54, 57, 60, 63, 66, 69)	(35, 36, 37, 71, 72, 73)	(-88, -70, -16, 2, 56, 74)
(54, 57, 60, 63, 66, 69)	(17, 18, 19, 71, 72, 73)	(-70, -52, -34, 38, 56, 74)
(54, 57, 60, 63, 66, 69)	(-16, 2, 20, 38, 56, 74)	(-37, -36, -35, 71, 72, 73)
(56, 57, 58, 68, 69, 70)	(23, 26, 47, 50, 71, 74)	(-78, -72, -6, 0, 66, 72)
(56, 57, 58, 68, 69, 70)	(18, 24, 42, 48, 66, 72)	(-73, -70, -1, 2, 71, 74)
(53, 55, 57, 65, 67, 69)	(24, 25, 48, 49, 72, 73)	(-76, -70, -4, 2, 68, 74)
(53, 55, 57, 65, 67, 69)	(20, 26, 44, 50, 68, 74)	(-72, -71, 0, 1, 72, 73)
(52, 53, 60, 61, 68, 69)	(23, 25, 47, 49, 71, 73)	(-74, -70, -2, 2, 70, 74)
(52, 53, 60, 61, 68, 69)	(22, 26, 46, 50, 70, 74)	(-73, -71, -1, 1, 71, 73)
(51, 53, 59, 61, 67, 69)	(24, 25, 48, 49, 72, 73)	(-74, -70, -2, 2, 70, 74)
(51, 53, 59, 61, 67, 69)	(22, 26, 46, 50, 70, 74)	(-72, -71, 0, 1, 72, 73)
(54, 58, 62, 66, 70, 74)	(20, 21, 44, 45, 68, 69)	(-73, -71, -1, 1, 71, 73)
(54, 58, 62, 66, 70, 74)	(19, 21, 43, 45, 67, 69)	(-72, -71, 0, 1, 72, 73)
(53, 54, 55, 71, 72, 73)	(27, 30, 33, 63, 66, 69)	(-79, -70, -7, 2, 65, 74)
(50, 51, 52, 68, 69, 70)	(24, 30, 36, 60, 66, 72)	(-73, -70, -1, 2, 71, 74)
(50, 51, 52, 68, 69, 70)	(-1, 2, 35, 38, 71, 74)	(-48, -42, -36, 60, 66, 72)
(53, 54, 55, 71, 72, 73)	(-7, 2, 29, 38, 65, 74)	(-45, -42, -39, 63, 66, 69)
(47, 49, 51, 65, 67, 69)	(26, 32, 38, 62, 68, 74)	(-72, -71, 0, 1, 72, 73)
(47, 49, 51, 65, 67, 69)	(0, 1, 36, 37, 72, 73)	(-46, -40, -34, 62, 68, 74)
(45, 48, 51, 63, 66, 69)	(35, 36, 37, 71, 72, 73)	(-79, -70, -7, 2, 65, 74)
(45, 48, 51, 63, 66, 69)	(-7, 2, 29, 38, 65, 74)	(-37, -36, -35, 71, 72, 73)
(44, 45, 56, 57, 68, 69)	(34, 36, 38, 70, 72, 74)	(-77, -71, -5, 1, 67, 73)
(44, 45, 56, 57, 68, 69)	(30, 34, 38, 66, 70, 74)	(-73, -71, -1, 1, 71, 73)
(44, 45, 56, 57, 68, 69)	(-1, 1, 35, 37, 71, 73)	(-42, -38, -34, 66, 70, 74)
(44, 45, 56, 57, 68, 69)	(-5, 1, 31, 37, 67, 73)	(-38, -36, -34, 70, 72, 74)
(43, 45, 55, 57, 67, 69)	(30, 34, 38, 66, 70, 74)	(-72, -71, 0, 1, 72, 73)
(43, 45, 55, 57, 67, 69)	(0, 1, 36, 37, 72, 73)	(-42, -38, -34, 66, 70, 74)
(42, 45, 54, 57, 66, 69)	(35, 36, 37, 71, 72, 73)	(-76, -70, -4, 2, 68, 74)
(42, 45, 54, 57, 66, 69)	(-4, 2, 32, 38, 68, 74)	(-37, -36, -35, 71, 72, 73)
(42, 48, 54, 60, 66, 72)	(32, 33, 34, 68, 69, 70)	(-73, -70, -1, 2, 71, 74)
(43, 49, 55, 61, 67, 73)	(-4, -3, 32, 33, 68, 69)	(-38, -36, -34, 70, 72, 74)
(42, 48, 54, 60, 66, 72)	(-1, 2, 35, 38, 71, 74)	(-40, -39, -38, 68, 69, 70)
(44, 45, 46, 71, 72, 73)	(9, 12, 15, 63, 66, 69)	(-52, -43, -34, 56, 65, 74)
(44, 45, 46, 71, 72, 73)	(2, 11, 20, 56, 65, 74)	(-45, -42, -39, 63, 66, 69)
(36, 39, 42, 63, 66, 69)	(17, 18, 19, 71, 72, 73)	(-52, -43, -34, 56, 65, 74)
(36, 39, 42, 63, 66, 69)	(2, 11, 20, 56, 65, 74)	(-37, -36, -35, 71, 72, 73)
(32, 33, 50, 51, 68, 69)	(16, 18, 20, 70, 72, 74)	(-47, -41, -35, 61, 67, 73)
(32, 33, 50, 51, 68, 69)	(7, 13, 19, 61, 67, 73)	(-38, -36, -34, 70, 72, 74)
(30, 33, 48, 51, 66, 69)	(17, 18, 19, 71, 72, 73)	(-46, -40, -34, 62, 68, 74)
(30, 33, 48, 51, 66, 69)	(8, 14, 20, 62, 68, 74)	(-37, -36, -35, 71, 72, 73)
(29, 38, 47, 56, 65, 74)	(17, 18, 19, 71, 72, 73)	(-45, -42, -39, 63, 66, 69)
(29, 38, 47, 56, 65, 74)	(9, 12, 15, 63, 66, 69)	(-37, -36, -35, 71, 72, 73)