Challenge Of the Week

March 4—March 10, 2008

Problem

Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person "A" phones person "B", "A" tells "B" everything he knows, while "B" tells "A" nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything?

Solution

The minimum number of calls for everyone to know everything is $2n - 2$.

Label the people $P_1, \ldots, P_n$. We first show that $2n - 2$ is a sufficient number of calls.

The following algorithm will make all information known to everyone after exactly $2n - 2$ calls: For the first $n - 1$ calls, each P_i $(i = 1, \ldots, n - 1)$ calls P_n. After this, P_n knows everything. In the next $n - 1$ calls, P_n calls each P_i for $i < n$.

Next, we show that $2n - 2$ is a necessary number of calls.

Suppose we have a sequence of calls which leaves everybody fully informed. Consider the "crucial" call at the end of which the receiver P_k is the first to be fully informed. Clearly, each of the $n - 1$ people other than P_k must have placed at least one call no later than the crucial call (how else could P_k be fully informed?). Also, each of these $n - 1$ people (being not fully informed) must receive at least one call after the crucial one. Hence the given sequence contains at least $2(n - 1)$ calls.