Challenge Of the Week

April 8—April 14, 2008

Problem:

For any integer k, define $f(k)$ to be the square of the sum of the digits of k (writing k in base 10, as usual!). Next, define $f_m(k) = f(f(f(\ldots f(k))))$, where the function f is applied m times. For example, $f_2(35) = f(f(35)) = f(64) = 100$.

The problem is to compute $f_{2008}(2^{2008})$.

Solution:

The answer is $f_{2008}(2^{2008}) = 169$. The fastest way to arrive at this is using a symbolic computer package, but this is the cheap way out. To get the answer by hand requires two kinds of analysis. The first gives a bound on the value, reducing the problem to manageable numbers; the second examines the "finer" details of the answer, which allows us to nail it down completely.

Macroscopic Analysis

Let $N = 2^{2008}$. Then we have $\log_{10}(N) = \log_2(N) \cdot \log_{10}(2) = (2008)(0.301) = 604.5$. So N has 605 digits. This means that the largest $f_1(N)$ can be is

$$f_1(N) \leq f(999\ldots999) = (605 \cdot 9)^2 = 29648025.$$

From this, we can compute bounds on the next few iterates:

$$\begin{aligned}
f_2(N) &\leq f(29999999) = (2 + 7 \cdot 9)^2 = 4225 \\
f_3(N) &\leq f(4999) = (4 + 3 \cdot 9)^2 = 961 \\
f_4(N) &\leq f(999) = (3 \cdot 9)^2 = 729 \\
f_5(N) &\leq f(999) = 729
\end{aligned}$$

So we find that $f_m(N) \leq 729$ for all $m \geq 4$. (We can establish better bounds if we're slightly more careful.)

Microscopic Analysis

By definition, $f_m(N)$ is a square number. Thus our list of possiblities drops to the square numbers $0, 1, 4, \ldots, 729$. To find out which one, we begin by working modulo 9. The motivation for this is that the sum of the digits of a number is the same as the number (mod 9); this is the basis for the basis for the usual divisibility test by 9. To see why this works, let $S(n)$ denote the sum of digits of n. If we write n in decimal as $n = \sum_i d_i 10^i$, then we get $n = \sum d_i 10^i \equiv \sum d_i \cdot 1^i = \sum d_i = S(n) \pmod 9$.

The key point here is that $f(n) = S(n)^2 \equiv n^2 \pmod 9$. To compute N modulo 9, note that $2^6 = 64 \equiv 1 \pmod 9$, so that

$$N = 2^{2008} = 2^{6 \cdot 334 + 4} \equiv 2^4 \equiv 7 \pmod 9.$$

Then, iterating, we find

$$\begin{aligned} f_1(N) &\equiv 7^2 \equiv 4 \pmod 9 \\ f_2(N) &\equiv 4^2 \equiv 7 \pmod 9 \\ f_3(N) &\equiv 7^2 \equiv 4 \pmod 9 \\ f_4(N) &\equiv 4^2 \equiv 7 \pmod 9 \\ &\vdots \end{aligned}$$

Now, the only candidates for $f_4(N)$ are the square numbers congruent to 7 that are less than or equal to 729. Specifically, $f_4(N) \in \{16, 25, 169, 196, 484, 529\}$.

Applying f to each of the six aboves numbers shows that $f_5(N)$ must be either 49 or 256. Applying f to 49 and 256 shows that $f_6(N) = 169$. At this point, there is no uncertainty! We know that $f_7(N) = 256$, and $f_8(N) = 169$ again. Thus all even $m \geq 6$ has $f_m(N) = 169$, so in particular, $f_{2008}(N) = 169$.