Challenge Of the Week

April 15—April 21, 2008

Problem:

Find the minimum number of squares that are required to draw a complete $n \times n$ grid. For example, when $n = 2$, the answer is 3, as three squares suffice: a 2×2 and two 1×1 squares (draw a picture!); moreover it is easy to see it can't be done with fewer.

Solution:

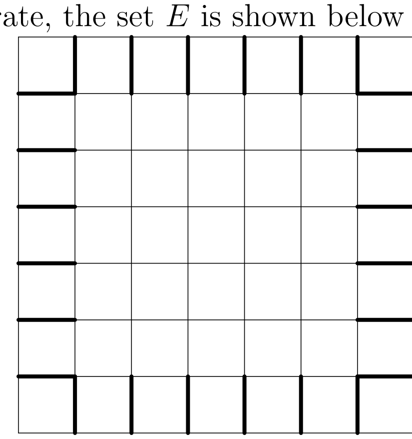
Let $f(n)$ denote the minimum number of squares needed to draw an $n \times n$ grid. We clearly have $f(1) = 1$ and $f(2) = 3$. For $n \geq 3$, the answer is $f(n) = 2(n - 1)$.

To prove this, we first argue that at least $2(n - 1)$ squares are necessary. For notation, suppose the $n \times n$ grid has coordinates (i, j), where $0 \leq i, j \leq n$.

Consider the set of $4(n - 1)$ line segments

$E = \{(i, 0) - (i, 1), \quad (n-1, i) - (n, i), \quad (0, i) - (1, i), \quad (i, n-1) - (i, n)\}$ for $1 \leq i \leq n-1$.

E consists of line segments of unit length around the border of the grid that are perpendicular to the outer square. To illustrate, the set E is shown below for the 7×7 grid:



Observe that any square can overlap at most two segments in E. Since every edge in E must be drawn as part of a complete grid, we must have at least

$$\frac{4(n-1) \text{ segments}}{2 \text{ segments/square}} = 2(n-1) \text{ squares.}$$

The second part of the proof is showing that $2(n-1)$ squares suffice. This is done by explicitly constructing the drawing.

We can think of the grid as being made up of $2n-2$ internal grid lines, $n-1$ horizontal and $n-1$ vertical, plus an outer square border. To begin construction of the $n \times n$ grid, anchor one $(n-1) \times (n-1)$ square to each of the four corners of the desired $n \times n$ grid. Observe that this achieves complete coverage of four internal grid lines (two horizontal and two vertical), along with the entire external border of the $n \times n$ grid. So far, we have used four squares to cover four internal grid lines, leaving $2n-6$ to go.

To draw the remaining internal grid lines, we can anchor $n-3$ pairs of squares having complementary side lengths (i.e., 2×2 and $(n-2) \times (n-2)$ form a pair, as do 3×3 and $(n-3) \times (n-3)$, etc.) at opposite corners of the $n \times n$ grid. Each pair of squares completely covers two distinct internal grid lines (one horizontal and one vertical), so that $n-3$ pairs, or $2(n-3) = 2n-6$ squares, completely cover the remaining internal grid lines.

In total, then, this construction uses $4 + (2n-6) = 2(n-1)$ squares to achieve complete coverage of all $2n-2$ internal grid lines, along with the external border, which entirely fills out the $n \times n$ grid. This completes the argument.