Challenge of the Week April 22–April 28, 2008

Problem

Andrew has n^3 white cubes of the size $1 \times 1 \times 1$. He wants to construct an $n \times n \times n$ cube with all its faces being completely white. Find the minimal number of the faces of small cubes that Basil must paint black in order to prevent Andrew from completing his task. Consider the cases $n = 3$ and $n = 1000$.

Solution

We'll give Basil's strategy for general n. When $n = 1$, obviously Basil just has to paint one of the sides. So, suppose that $n \geq 2$ from here on.

A full $n \times n \times n$ cube consists of $(n-2)^3$ interior cubes, $6(n-2)^2$ face cubes, $12(n-2)$ edge cubes, and 8 corner cubes. Basil can foil Andrew in one of three ways: forcing a black side to show up on a corner cube, an edge cube, or a face cube.

Corner cubes: If Basil paints two opposite faces of a cube black, one of the black faces must appear if the cube is placed in the corner. So Basil just paints enough cubes so that Andrew has no choice but to put a painted cube in the corner. He needs to paint opposite sides on

$$\underbrace{(n-2)^3}_{\text{interior}} + \underbrace{6(n-2)^2}_{\text{face}} + \underbrace{12(n-2)}_{\text{edge}} + \underbrace{1}_{\text{corner}} = n^3 - 7$$

cubes, involving $2(n^3 - 7)$ faces.

Edge cubes: If Basil paints two pairs of opposite sides of a cube black, then a black side must show when the cube is on the edge (or corner). So Basil can paint 4 sides of just enough cubes to force poor Andrew to put one on an edge (or corner). Here, he needs

$$\underbrace{(n-2)^3}_{\text{interior}} + \underbrace{6(n-2)^2}_{\text{face}} + \underbrace{1}_{\text{edge or corner}} = n^3 - 12n + 17$$

cubes, involving $4(n^3 - 12n + 17)$ faces.

Face cubes: If Basil paints all 6 sides of a cube black, then at least one black side must show if the cube is on the face (or edge, or corner). Here, Basil just has to paint $(n-2)^3 + 1$ cubes, all the interior plus one more, frustrating Andrew yet again with $6((n-2)^3 + 1)$ black faces.

As Basil wants to minimize the amount of painting he has to do, he just takes the minimum of $2(n^3 - 7)$, $4(n^3 - 12n + 17)$, and $6((n-2)^3 + 1)$ for any given n. It's easy to check that the best Basil can do is to adopt the corner cube strategy ($2(n^3 - 7)$ faces) for $n = 2$ and $n \geq 7$, and to use the face cube strategy ($6((n-2)^3 - 1)$ faces) for $n = 3, 4, 5$ and 6. In particular, Basil needs 12 faces when $n = 3$ and 1999999986 faces when $n = 1000$.