

## Challenge Of the Week

April 29—May 5, 2008

### Problem:

Let  $ABCD$  be a unit square. Suppose  $E$  is a point on  $AB$  and  $F$  is a point on  $AD$  so that the triangle  $AEF$  has perimeter 2. Find angle  $ECF$ .

### Solution:

**Solution 1:** The wording of the question implies that angle  $ECF$  is independent of the exact locations of  $E$  and  $F$ . If we assume this, then we may suppose that  $E$  is at  $B$  and  $F$  is at  $A$ . In this case  $ECF$  is  $45^\circ$ , so this must be the answer.

**Solution 2:** Construct a new point  $G$  on the line  $AB$ , outside the square, so that  $|BG| = |DF|$ . For convenience, let  $x = |AE|$  and  $y = |AF|$ , so that  $|EB| = 1 - x$ ,  $|FD| = 1 - y$ ,  $|BG| = 1 - y$ . Now, note that triangles  $CDF$  and  $CBG$  are congruent since they are both right triangles with leg lengths 1 and  $1 - y$ . So  $|FC| = |CG|$ . Also, note that  $|FE| = 2 - x - y = |EG|$ . Since  $|FC| = |CG|$  and  $|FE| = |EG|$ , the triangles  $FCE$  and  $GCE$  are congruent, and so angle  $FCE$  equals  $GCE$  equals half of angle  $FCG$ . Since angle  $FCG = 90^\circ$ , we have angle  $ECF = 45^\circ$ .

