Challenge Of the Week

May 6—May 12, 2008

Problem

In an old edition of Ripley’s Believe It Or Not, it was stated that the number

\[ N = 526315789473684210 \]

is a "persistent" number. That is, if multiplied by any positive integer the resulting number always contains the ten digits 0, 1, 2, \ldots, 9 in some order with possible repetitions.

(a) Prove or disprove the above statement.

(b) Are there any persistent numbers smaller than the above number \( N \)?

Solution

The statement (a) is false. To see this, note that \( 2N = 1052631578947368420 \), so that \( 19N = 20N - N = 9999999999999999990 \), which only contains two digits!

The answer to (b) is “no.” In fact there are no persistent numbers!

Proof 1: (Due to Jacob Lewis)

To show an arbitrary integer \( N \) is not persistent, examine the decimal expansion of \( 1/N \):

\[
\frac{1}{N} = 0.a_1a_2\ldots a_r b_1b_2\ldots b_s \\
= \frac{a_1a_2\ldots a_r}{10^r} + \frac{b_1b_2\ldots b_s}{10^r(10^s - 1)} \\
= \frac{(10^s - 1)a_1a_2\ldots a_r + b_1b_2\ldots b_s}{10^r(10^s - 1)}
\]

Cross multiplying, we get

\[
10^r(10^s - 1) = ((10^s - 1)a_1a_2\ldots a_r + b_1b_2\ldots b_s)N
\]

The left side consists only of 0’s and 9’s, while the right is a multiple of \( N \). Thus \( N \) is not persistent. (N.B.: if \( 1/N \) has no repeating part, the same argument, suitably simplified, still works, but the left side has the form \( 10^r \) which only involves 0’s and 1’s.)
Proof 2: Assume again that \( N \) is persistent. Consider the remainders obtained by dividing the numbers
\[
1, 11, 111, \ldots, \underbrace{111\ldots1}_{N \text{ ones}}
\]
by \( N \). At most \( N - 1 \) different nonzero remainders can result, either one of the above numbers is divisible by \( N \), in which case \( N \) is not persistent, or else two of them, say
\[
R = \underbrace{111\ldots1}_{r} \quad \text{and} \quad S = \underbrace{111\ldots1}_{s}
\]
give the same remainder. Without loss of generality, assume \( S > R \), and so their difference
\[
S - R = \underbrace{111\ldots1000\ldots0}_{s-r}
\]
is divisible by \( N \), and \( N \) is not persistent.

Proof 3: This uses Euler’s generalization of Fermat’s Little Theorem. Define \( \varphi(n) \) to be the number of positive integers less than or equal to \( n \) that are relatively prime to \( n \). (E.g., if \( p \) is prime, the \( \varphi(p) = p - 1 \).) Euler’s theorem is the following: If \( a \) is relatively prime to \( n \), then \( a^{\varphi(n)} - 1 \) is a multiple of \( n \).

Assume that \( N \) is persistent. We can express \( N \) in the form \( N = 2^a5^bM \), where \( M \) is relatively prime to both 2 and 5. Since \( 2^b5^aN = 10^a + bM \) must also be persistent, all multiples of \( M \) must contain the nine nonzero digits. Using Euler’s theorem, we can write
\[
10^{\varphi(M)} - 1 = kM
\]
for some \( k \). This gives a contradiction, for \( kM \) should contain all nine nonzero digits, but \( 10^{\varphi(M)} - 1 \) contains only nines.