Challenge Of the Week

May 6—May 12, 2008

Problem

In an old edition of Ripley's Believe It Or Not, it was stated that the number

$$N = 526315789473684210$$

is a "persistent" number. That is, if multiplied by any positive integer the resulting number always contains the ten digits $0, 1, 2, \ldots, 9$ in some order with possible repetitions.

(a) Prove or disprove the above statement.

(b) Are there any peristent numbers smaller than the above number N?

Solution

The statement (a) is false. To see this, note that $2N = 1052631578947368420$, so that $19N = 20N - N = 9999999999999999990$, which only contains two digits!

The answer to (b) is "no." In fact there are no persistent numbers!

Proof 1: (Due to Jacob Lewis)

To show an arbitrary integer N is not persistent, examine the decimal expansion of $1/N$:

$$\begin{aligned} \frac{1}{N} &= 0.a_1 a_2 \ldots a_r \overline{b_1 b_2 \ldots b_s} \\ &= \frac{a_1 a_2 \ldots a_r}{10^r} + \frac{b_1 b_2 \ldots b_s}{10^r(10^s - 1)} \\ &= \frac{(10^s - 1)a_1 a_2 \ldots a_r + b_1 b_2 \ldots b_s}{10^r(10^s - 1)} \end{aligned}$$

Cross multiplying, we get

$$10^r(10^s - 1) = ((10^s - 1)a_1 a_2 \ldots a_r + b_1 b_2 \ldots b_s)N$$

The left side consists only of 0's and 9's, while the right is a multiple of N. Thus N is not persistent. (N.B.: if $1/N$ has no repeating part, the same argument, suitably simplified, still works, but the left side has the form 10^r which only involves 0's and 1's.)

Proof 2: Assume again that N is persistent. Consider the remainders obtained by dividing the numbers

$$1, 11, 111, \ldots, \underbrace{111\ldots1}_{N \text{ ones}}$$

by N. At most $N-1$ different nonzero remainders can result, either one of the above numbers is divisible by N, in which case N is not persistent, or else two of them, say

$$R = \underbrace{111\ldots1}_{r} \quad \text{and} \quad S = \underbrace{111\ldots1}_{s}$$

give the same remainder. Without loss of generality, assume $S > R$, and so their difference

$$S - R = \underbrace{111\ldots1}_{s-r}\underbrace{000\ldots0}_{r}$$

is divisible by N, and N is not persistent.

Proof 3: This uses Euler's generalization of Fermat's Little Theorem. Define $\varphi(n)$ to be the number of positive integers less than or equal to n that are relatively prime to n. (E.g., if p is prime, the $\varphi(p) = p - 1$.) Euler's theorem is the following: *If a is relatively prime to n, then $a^{\varphi(n)} - 1$ is a multiple of n.*

Assume that N is persistent. We can express N in the form $N = 2^a 5^b M$, where M is relatively prime to both 2 and 5. Since $2^b 5^a N = 10^{a+b} M$ must also be persistent, all multiples of M must contain the nine nonzero digits. Using Euler's theorem, we can write

$$10^{\varphi(M)} - 1 = kM$$

for some k. This gives a contradiction, for kM should contain all nine nonzero digits, but $10^{\varphi(M)} - 1$ contains only nines.