Challenge Of the Week

May 20—May 26, 2008

Problem:

Suppose we have a room with infinitely many boxes, and a child, Joe, who likes to shuffle his toys among these boxes. Initially, there is one box containing AB toys, while the rest are empty. Every hour Joe takes one toy from each non-empty box and put these toys (together) in an empty box. At some time there are A boxes each holding B toys.

For which A and B is this possible?

Solution:

(by Robert Bradshaw)

The possibilities are given by:

- $A = 1$, B arbitrary (trivial, starts as a solution)
- $A = 0$ or $B = 0$ (trivial, starts as a solution)
- $A = 2$, $B = 1$
- $A = 2$, $B = 2$

Lemma. At any point there are at most two non-empty boxes with the same number of items in them, i.e. $A \leq 2$.

Let the initial box start with n toys, $n > 1$.

First I claim the number of toys in each of the non-initial non-empty boxes are all distinct and bounded by the number of non-empty boxes for $k < n$. This is vacuously satisfied at the start. Assume this holds at step k, and let $\mathsf{boxes}(k)$ be the number of non-empty boxes. Now $\mathsf{boxes}(k+1) \geq \mathsf{boxes}(k)$ because by uniqueness we empty at most one box (the box with one toy, if it exists) and fill one previously-empty box. Because the count at step k is bounded by $\mathsf{boxes}(k)$, the count for each of these at $k+1$ is bounded by $\mathsf{boxes}(k) - 1$, which is strictly bounded by $\mathsf{boxes}(k)$, the number of items in the newly filled box, so uniqueness is preserved as well. This shows the lemma for $k < n$.

Now I can show that the lemma holds for $k \geq n-1$. Also the number of items in any box at step k is bounded by $\mathsf{boxes}(k)$ unless step $k-1$ has two boxes with one item each. By the above claim, this holds true at $k = n-1$. If, at step k, there is at most one box with one item, then as before $\mathsf{boxes}(k+1) \geq \mathsf{boxes}(k)$ and hence the newly filled box has more items than all other boxes. If, however there are two boxes with one item, then $\mathsf{boxes}(k+1) = \mathsf{boxes}(k) - 1$, so all boxes at step $k+1$ are unique but the bound does not hold. In this case, at step $k+2$, the bound is once again satisfied but the new box may have as many items as the previous new box (which is okay because of uniqueness at step k+1). $\square$

Now, suppose that $A = 2$, the maximum allowed by the lemma. If at step k we wish to have A boxes with B toys each, then at step $k-1$ we need exactly one box with $B+1$ toys (for the first of the two boxes), and $B-1$ boxes with a single toy each (which will give us the other, and all others becoming empty). By the lemma above, $B - 1 \leq 2$, and trying out $B = 3$ (six initial toys) does not work, but $B = 1$ and $B = 2$ do, giving the only non-trivial solutions.

Note

Most people approached this by just working forward given a particular number of starting toys. It's easy to see (though tedious to prove formally) how the numbers of toys in the boxes are intimately related to triangular numbers.

For example, let $[a_1, a_2, \ldots, a_m]$ denote the state where we have m nonempty boxes, with a_i toys in box i. If we start with n toys, then the sequence of boxes/toys is

$$\begin{aligned}
[n] &\to [n-1,\ 1] \\
&\to [n-2,\ 2] \to [n-3,\ 1,2] \\
&\to [n-4,\ 1,2] \to [n-5,\ 2,3] \to [n-6,\ 1,2,3] \\
&\to \quad \ldots \quad \to [n-10,\ 1,2,3,4] \\
&\to \quad \ldots \quad \to [n-15,\ 1,2,3,4,5] \\
&\vdots
\end{aligned}$$