

## Challenge of the Week

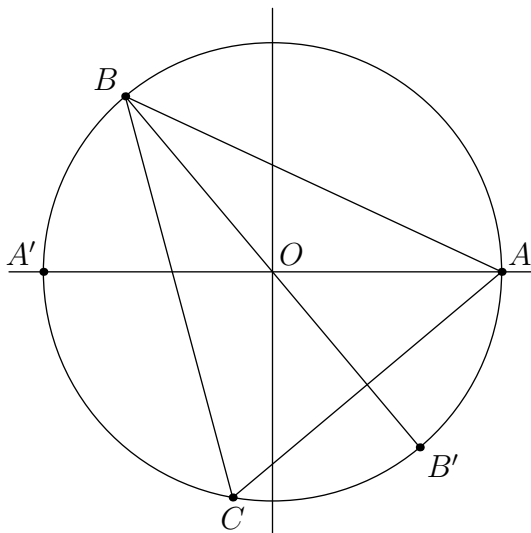
May 27—June 3, 2008

### Problem

Let three points be chosen at random, uniformly with respect to arc length, on the circumference of a given circle. Determine the probability that the triangle determined by the three points is acute.

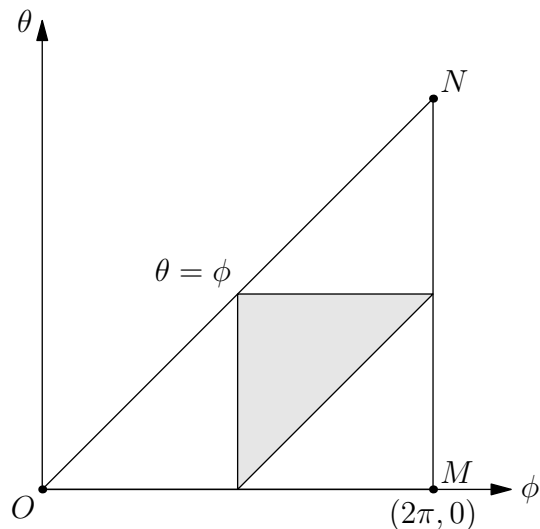
### Solution 1

Let the given circle have center  $O$ , and let the three random points be  $A$ ,  $B$ , and  $C$ . Without loss of generality, we can assume that  $A$  is fixed on the  $x$ -axis. Let  $\angle AOB = \theta$  and  $\angle AOC = \phi$ , measured counterclockwise from  $AO$ . Then both  $\theta$  and  $\phi$  fall in the interval  $[0, 2\pi]$ .  $\theta = \phi$  with probability zero, and the cases  $\theta > \phi$  and  $\theta < \phi$  are equally probable. So we'll assume that  $\phi > \theta$ , as shown below.



Let  $A'$  and  $B'$  be the points opposite  $A$  and  $B$  on the circle. For  $\triangle ABC$  to be acute,  $\theta < \pi$ , and  $C$  must lie on arc  $A'B'$ . (If  $C$  is between  $B$  and  $A'$ , then  $\angle CBA$  would be obtuse; if  $C$  is between  $B'$  and  $A$ , then  $\angle BAC$  would be obtuse.) The set of restrictions for an acute triangle require  $\pi < \phi < \theta + \pi$ .

With this setup, let's look at the probability of getting an acute triangle in terms of the angles  $\phi$  and  $\theta$ .  $(\phi, \theta)$  come from the uniform probability space  $\{(\phi, \theta) \mid 0 < \phi < 2\pi, 0 < \theta < \phi\}$ . I.e., they are distributed uniformly in the triangle  $\triangle OMN$  in the following figure:



The set of restrictions for an acute triangle require  $(\phi, \theta)$  to lie on the triangle whose vertices are the midpoints of  $ON$ ,  $NM$ , and  $MO$ , a triangle with area one-quarter that of  $\triangle OMN$ , hence the required probability is  $1/4$ .

### Solution 2

Assume without loss of generality that the circumference of the circle is 1. Let  $t$  be the shorter arc length between the points  $A$  and  $B$ ;  $t$  is chosen uniformly in the interval  $[0, 1/2]$ . If  $\triangle ABC$  is to be acute, then  $C$  must fall in the the arc of length  $t$  between  $A'$  and  $B'$ . This occurs with probability  $t$ . Thus the probability that  $\triangle ABC$  is acute is

$$\frac{1}{1/2} \int_0^{1/2} t dt = \frac{1}{4}.$$

