A Proof of the Sum of Squares Conjecture

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We present a proof of the below conjecture appearing as a problem of the week for the week for Jun 3, 2008 at http://www.math.washington.edu/∼challenge/.

The conjecture:
Any positive integer n can be written in the form $n = \sum_{k=1}^{m} a_k k^2$ where $|a_k| = 1$.

In fact, we show that there are an infinite number of such representations. First, we will prove the following Lemma which we will need:

Lemma: *If $m \geq 12$ is an integer, and if N is any integer satisfying*

$$128 < N < \frac{\sum_{i=1}^{m} i^2}{2} \tag{1}$$

then N can be written as a sum of distinct perfect squares as

$$N = \sum_{i=1}^{k} x_i^2, |x_i| \neq |x_j|, 0 < |x_i| \leq m \tag{2}$$

Proof: We proceed by induction on m. The case when $m = 12$ was verified with the aid of a computer. See appendix. Suppose now, the hypothesis is true for $m = n - 1$ and we need to show that it is true for $m = n$.
Assume we are given an integer N for which

$$128 < N < \frac{\sum_{i=1}^{n} i^2}{2} \tag{3}$$

Now if $2N < \sum_{i=1}^{n-1} i^2$, then by induction hypothesis, we are done. So assume that $2N \geq \sum_{i=1}^{n-1} i^2$. In which case, consider $N' = N - n^2$. We can show that for

$n \geq 13$, N' satisifies

$$128 < N' < \frac{\sum_{i=1}^{n-1} i^2}{2} \tag{4}$$

That $N' < RHS$ of equation 4 is easy to see. To show $128 < N'$, consider

$$N' \geq \frac{\sum_{i=1}^{n-1} i^2}{2} - n^2 = \frac{2n^3 - 15n^2 + n}{12} = f(n) \tag{5}$$

We can easily see that $f(n)$ is an increasing function of n for $n \geq 10$ and that $f(13) = 156 > 128$.

We can now apply the induction hypothesis to N' and can safely add n^2 to the resulting representation of N' without violating the distinctness property to get the required representation for N. Thus, the lemma is true for all $m \geq 12$. $\diamond$

Now we apply the above lemma to prove the conjecture. Given a positive integer s consider an integer N, for which, $2N = \sum_{i=1}^{m} i^2 - s$. For sufficiently large m (and for an infinite number of such m) we have the N satisifies equation 1 of the above lemma and so can be written in the form as in equation 2 of the lemma. Thus we have that $N = \sum_{i=1}^{k} x_i^2, |x_i| \neq |x_j|, 0 < |x_i| \leq m$. Thus we have that $s = \sum_{i=1}^{m} i^2 - 2\sum_{i=1}^{k} x_i^2$ which proves the conjecture, as the coefficient of x_i^2 will be -1 and the other coefficients that appear will be 1, when s is re-written as $s = \sum_{i=1}^{m} a_i i^2$.

Appendix

Here we show that the base case $m = 12$ of the lemma appearing above is true by giving a listing of the required representation for N between 128 and 326 (non-inclusive). The list below was generated (and verified) with the aid of a computer.

$129 = 10^2 + 5^2 + 2^2$	$130 = 11^2 + 3^2$
$131 = 11^2 + 3^2 + 1^2$	$132 = 9^2 + 5^2 + 4^2 + 3^2 + 1^2$
$133 = 9^2 + 6^2 + 4^2$	$134 = 11^2 + 3^2 + 2^2$
$135 = 11^2 + 3^2 + 2^2 + 1^2$	$136 = 10^2 + 6^2$
$137 = 11^2 + 4^2$	$138 = 11^2 + 4^2 + 1^2$

$139 = 10^2 + 5^2 + 3^2 + 2^2 + 1^2$

$140 = 10^2 + 6^2 + 2^2$

$141 = 11^2 + 4^2 + 2^2$

$142 = 11^2 + 4^2 + 2^2 + 1^2$

$143 = 9^2 + 7^2 + 3^2 + 2^2$

$144 = 12^2$

$145 = 12^2 + 1^2$

$146 = 11^2 + 5^2$

$147 = 11^2 + 5^2 + 1^2$

$148 = 12^2 + 2^2$

$149 = 12^2 + 2^2 + 1^2$

$150 = 11^2 + 5^2 + 2^2$

$151 = 11^2 + 5^2 + 2^2 + 1^2$

$152 = 10^2 + 6^2 + 4^2$

$153 = 12^2 + 3^2$

$154 = 12^2 + 3^2 + 1^2$

$155 = 11^2 + 5^2 + 3^2$

$156 = 11^2 + 5^2 + 3^2 + 1^2$

$157 = 12^2 + 3^2 + 2^2$

$158 = 12^2 + 3^2 + 2^2 + 1^2$

$159 = 11^2 + 5^2 + 3^2 + 2^2$

$160 = 12^2 + 4^2$

$161 = 12^2 + 4^2 + 1^2$

$162 = 11^2 + 6^2 + 2^2 + 1^2$

$163 = 11^2 + 5^2 + 4^2 + 1^2$

$164 = 12^2 + 4^2 + 2^2$

$165 = 12^2 + 4^2 + 2^2 + 1^2$

$166 = 11^2 + 6^2 + 3^2$

$167 = 11^2 + 6^2 + 3^2 + 1^2$

$168 = 10^2 + 8^2 + 2^2$

$169 = 12^2 + 5^2$

$170 = 12^2 + 5^2 + 1^2$

$171 = 11^2 + 7^2 + 1^2$

$172 = 11^2 + 5^2 + 4^2 + 3^2 + 1^2$

$173 = 12^2 + 5^2 + 2^2$

$174 = 12^2 + 5^2 + 2^2 + 1^2$

$175 = 11^2 + 7^2 + 2^2 + 1^2$

$176 = 11^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2$

$177 = 11^2 + 6^2 + 4^2 + 2^2$

$178 = 12^2 + 5^2 + 3^2$

$179 = 12^2 + 5^2 + 3^2 + 1^2$

$180 = 12^2 + 6^2$

$181 = 12^2 + 6^2 + 1^2$

$182 = 12^2 + 5^2 + 3^2 + 2^2$

$183 = 12^2 + 5^2 + 3^2 + 2^2 + 1^2$

$184 = 12^2 + 6^2 + 2^2$

$185 = 12^2 + 6^2 + 2^2 + 1^2$

$186 = 12^2 + 5^2 + 4^2 + 1^2$

$187 = 11^2 + 7^2 + 4^2 + 1^2$

$188 = 10^2 + 7^2 + 5^2 + 3^2 + 2^2 + 1^2$

$189 = 12^2 + 6^2 + 3^2$

$190 = 12^2 + 6^2 + 3^2 + 1^2$

$191 = 11^2 + 7^2 + 4^2 + 2^2 + 1^2$

$192 = 11^2 + 6^2 + 5^2 + 3^2 + 1^2$

$193 = 12^2 + 7^2$

$194 = 12^2 + 7^2 + 1^2$

$195 = 12^2 + 5^2 + 4^2 + 3^2 + 1^2$

$196 = 12^2 + 6^2 + 4^2$

$197 = 12^2 + 7^2 + 2^2$

$198 = 12^2 + 7^2 + 2^2 + 1^2$

$199 = 12^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2$

$200 = 12^2 + 6^2 + 4^2 + 2^2$

$201 = 12^2 + 6^2 + 4^2 + 2^2 + 1^2$

$202 = 12^2 + 7^2 + 3^2$

$203 = 12^2 + 7^2 + 3^2 + 1^2$

$204 = 11^2 + 7^2 + 5^2 + 3^2$

$205 = 12^2 + 6^2 + 5^2$

$206 = 12^2 + 7^2 + 3^2 + 2^2$

$207 = 12^2 + 7^2 + 3^2 + 2^2 + 1^2$

$208 = 12^2 + 8^2$

$209 = 12^2 + 8^2 + 1^2$

$210 = 12^2 + 7^2 + 4^2 + 1^2$

$211 = 11^2 + 9^2 + 3^2$

$212 = 12^2 + 8^2 + 2^2$

$213 = 12^2 + 8^2 + 2^2 + 1^2$

$214 = 12^2 + 7^2 + 4^2 + 2^2 + 1^2$

$215 = 12^2 + 6^2 + 5^2 + 3^2 + 1^2$

$216 = 11^2 + 9^2 + 3^2 + 2^2 + 1^2$

$217 = 12^2 + 8^2 + 3^2$

$218 = 12^2 + 8^2 + 3^2 + 1^2$

$219 = 12^2 + 7^2 + 5^2 + 1^2$

$220 = 11^2 + 8^2 + 5^2 + 3^2 + 1^2$

$221 = 12^2 + 8^2 + 3^2 + 2^2$

$222 = 12^2 + 8^2 + 3^2 + 2^2 + 1^2$

$223 = 12^2 + 7^2 + 5^2 + 2^2 + 1^2$

$224 = 12^2 + 8^2 + 4^2$

$225 = 12^2 + 9^2$

$226 = 12^2 + 9^2 + 1^2$

$227 = 12^2 + 7^2 + 5^2 + 3^2$

$228 = 12^2 + 8^2 + 4^2 + 2^2$

$229 = 12^2 + 9^2 + 2^2$

$230 = 12^2 + 9^2 + 2^2 + 1^2$

$231 = 12^2 + 7^2 + 5^2 + 3^2 + 2^2$

$232 = 12^2 + 7^2 + 5^2 + 3^2 + 2^2 + 1^2$

$233 = 12^2 + 8^2 + 5^2$

$234 = 12^2 + 9^2 + 3^2$

$235 = 12^2 + 9^2 + 3^2 + 1^2$

$236 = 11^2 + 9^2 + 5^2 + 3^2$

$237 = 12^2 + 8^2 + 5^2 + 2^2$

$238 = 12^2 + 9^2 + 3^2 + 2^2$

$239 = 12^2 + 9^2 + 3^2 + 2^2 + 1^2$

$240 = 11^2 + 9^2 + 5^2 + 3^2 + 2^2$

$241 = 12^2 + 9^2 + 4^2$

$242 = 12^2 + 9^2 + 4^2 + 1^2$

$243 = 12^2 + 8^2 + 5^2 + 3^2 + 1^2$

$244 = 12^2 + 10^2$

$245 = 12^2 + 10^2 + 1^2$

$246 = 12^2 + 9^2 + 4^2 + 2^2 + 1^2$

$247 = 12^2 + 8^2 + 5^2 + 3^2 + 2^2 + 1^2$

$248 = 12^2 + 10^2 + 2^2$

$249 = 12^2 + 10^2 + 2^2 + 1^2$

$250 = 12^2 + 9^2 + 5^2$

$251 = 12^2 + 9^2 + 5^2 + 1^2$

$252 = 11^2 + 9^2 + 7^2 + 1^2$

$253 = 12^2 + 10^2 + 3^2$

$254 = 12^2 + 10^2 + 3^2 + 1^2$

$255 = 12^2 + 9^2 + 5^2 + 2^2 + 1^2$

$256 = 11^2 + 10^2 + 5^2 + 3^2 + 1^2$

$257 = 12^2 + 10^2 + 3^2 + 2^2$

$258 = 12^2 + 10^2 + 3^2 + 2^2 + 1^2$

$259 = 12^2 + 9^2 + 5^2 + 3^2$

$260 = 12^2 + 10^2 + 4^2$

$261 = 12^2 + 10^2 + 4^2 + 1^2$

$262 = 12^2 + 9^2 + 6^2 + 1^2$

$263 = 12^2 + 9^2 + 5^2 + 3^2 + 2^2$

$264 = 12^2 + 10^2 + 4^2 + 2^2$

$265 = 12^2 + 11^2$

$266 = 12^2 + 11^2 + 1^2$

$267 = 12^2 + 9^2 + 5^2 + 4^2 + 1^2$

$268 = 12^2 + 7^2 + 6^2 + 5^2 + 3^2 + 2^2 + 1^2$

$269 = 12^2 + 11^2 + 2^2$

$270 = 12^2 + 11^2 + 2^2 + 1^2$

$271 = 12^2 + 9^2 + 6^2 + 3^2 + 1^2$

$272 = 11^2 + 10^2 + 5^2 + 4^2 + 3^2 + 1^2$

$273 = 12^2 + 10^2 + 5^2 + 2^2$

$274 = 12^2 + 11^2 + 3^2$

$275 = 12^2 + 11^2 + 3^2 + 1^2$

$276 = 12^2 + 9^2 + 5^2 + 4^2 + 3^2 + 1^2$

$277 = 12^2 + 9^2 + 6^2 + 4^2$

$278 = 12^2 + 11^2 + 3^2 + 2^2$

$279 = 12^2 + 11^2 + 3^2 + 2^2 + 1^2$

$280 = 12^2 + 10^2 + 6^2$

$281 = 12^2 + 11^2 + 4^2$

$282 = 12^2 + 11^2 + 4^2 + 1^2$

$283 = 12^2 + 10^2 + 5^2 + 3^2 + 2^2 + 1^2$

$284 = 12^2 + 10^2 + 6^2 + 2^2$

$285 = 12^2 + 11^2 + 4^2 + 2^2$

$286 = 12^2 + 11^2 + 4^2 + 2^2 + 1^2$

$287 = 12^2 + 9^2 + 7^2 + 3^2 + 2^2$

$288 = 12^2 + 9^2 + 7^2 + 3^2 + 2^2 + 1^2$

$289 = 12^2 + 10^2 + 6^2 + 3^2$

$290 = 12^2 + 11^2 + 5^2$

$291 = 12^2 + 11^2 + 5^2 + 1^2$

$292 = 12^2 + 8^2 + 7^2 + 5^2 + 3^2 + 1^2$

$293 = 12^2 + 10^2 + 7^2$

$294 = 12^2 + 11^2 + 5^2 + 2^2$

$295 = 12^2 + 11^2 + 5^2 + 2^2 + 1^2$

$296 = 12^2 + 10^2 + 6^2 + 4^2$

$297 = 12^2 + 10^2 + 7^2 + 2^2$

$298 = 12^2 + 10^2 + 7^2 + 2^2 + 1^2$

$299 = 12^2 + 11^2 + 5^2 + 3^2$

$300 = 12^2 + 11^2 + 5^2 + 3^2 + 1^2$

$301 = 12^2 + 11^2 + 6^2$

$302 = 12^2 + 11^2 + 6^2 + 1^2$

$303 = 12^2 + 11^2 + 5^2 + 3^2 + 2^2$

$304 = 12^2 + 11^2 + 5^2 + 3^2 + 2^2 + 1^2$

$305 = 12^2 + 11^2 + 6^2 + 2^2$

$306 = 12^2 + 11^2 + 6^2 + 2^2 + 1^2$

$307 = 12^2 + 11^2 + 5^2 + 4^2 + 1^2$

$308 = 12^2 + 10^2 + 8^2$

$309 = 12^2 + 10^2 + 8^2 + 1^2$

$310 = 12^2 + 11^2 + 6^2 + 3^2$

$311 = 12^2 + 11^2 + 6^2 + 3^2 + 1^2$

$312 = 12^2 + 10^2 + 8^2 + 2^2$

$313 = 12^2 + 10^2 + 8^2 + 2^2 + 1^2$

$314 = 12^2 + 11^2 + 7^2$

$315 = 12^2 + 11^2 + 7^2 + 1^2$

$316 = 12^2 + 11^2 + 5^2 + 4^2 + 3^2 + 1^2$

$317 = 12^2 + 11^2 + 6^2 + 4^2$

$318 = 12^2 + 11^2 + 7^2 + 2^2$

$319 = 12^2 + 11^2 + 7^2 + 2^2 + 1^2$

$320 = 12^2 + 11^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2$

$321 = 12^2 + 11^2 + 6^2 + 4^2 + 2^2$

$322 = 12^2 + 11^2 + 6^2 + 4^2 + 2^2 + 1^2$

$323 = 12^2 + 11^2 + 7^2 + 3^2$

$325 = 12^2 + 10^2 + 9^2$

$324 = 12^2 + 11^2 + 7^2 + 3^2 + 1^2$