Challenge of the Week

June 3—June 9, 2008

Problem

Observe that

$$\begin{aligned}
1 &= +1^2 \\
2 &= -1^2 - 2^2 - 3^2 + 4^2 \\
3 &= -1^2 + 2^2 \\
4 &= -1^2 - 2^2 + 3^2 \\
5 &= +1^2 + 2^2 \\
6 &= +1^2 - 2^2 + 3^2.
\end{aligned}$$

This suggests the conjecture: any positive integer n can be expressed in the form

$$n = e_1 1^2 + e_2 2^2 + ... + e_m m^2,$$

with m a positive integer, and $e_i = 1$ or -1, for $i = 1, 2, \ldots, m$.

Prove this conjecture.

Solution

The difference of consecutive squares from an arithmetic proression:

$$1^2 - 0^2 = 1, \qquad 2^2 - 1^2 = 3, \qquad 3^2 - 2^2 = 5, \qquad 4^2 - 3^2 = 7, \ldots$$

Taking every other term we form two arithmetic progressions, both with common difference 4:

$$\begin{aligned}
&1^2 - 0^2 = 1, \quad 3^2 - 2^2 = 5, \quad 5^2 - 4^2 = 9, \ldots \\
&2^2 - 1^2 = 3, \quad 4^2 - 3^2 = 7, \quad 6^2 - 5^2 = 11, \ldots
\end{aligned}$$

Explicitly writing out the common difference for either of these sequences we get the identity

$$\left((q+3)^2 - (q+2)^2\right) - \left((q+1)^2 - q^2\right) = 4, \tag{$*$}$$

which holds for any integer q.

This identity provides an inductive way to write out any integer n in the conjectured form. The problem statement shows how this may be done for the base cases $n = 1, 2, 3,$ and 4. For the inductive step, suppose we can write $n = e_1 1^2 + e_2 2^2 + ... + e_m m^2$. Then using identity $(*)$ with $q = m + 1$, we can write $n + 4$ as

$$n + 4 = e_1 1^2 + e_2 2^2 + ... + e_m m^2 + (m + 1)^2 - (m + 2)^2 - (m + 3)^2 + (m + 4)^2.$$

□

Note that while this construction gives one way to write any desired number as the sum or difference of the first m squares, it probably is not the most "efficient" way to do so. For example, to write 14, this construction (start at 2, apply 3 inductive steps) would give:

$$\begin{aligned} 14 &= 2 + 4 + 4 + 4 \\ &= (-1^2 - 2^2 - 3^2 + 4^2) + (5^2 - 6^2 - 7^2 + 8^2) + (9^2 - 10^2 - 11^2 + 12^2) + (13^2 - 14^2 - 15^2 + 16^2). \end{aligned}$$

Whereas there's a much simpler way:

$$14 = 1^2 + 2^2 + 3^2.$$