Challenge of the Week

June 10–June 23, 2008

Problem:

Determine A_{2008} if $A_0 = A$ and

$$A_{n+1} = \frac{A_n}{1 + nA_n}, \qquad \text{for } n = 0, 1, 2, \ldots$$

Solution:

Observe that $\frac{1}{A_{n+1}} - \frac{1}{A_n} = n$. Thus,

$$\begin{aligned} \frac{1}{A_1} - \frac{1}{A_0} &= 0, \\ \frac{1}{A_2} - \frac{1}{A_1} &= 1, \\ &\vdots \\ \frac{1}{A_{2008}} - \frac{1}{A_{2007}} &= 2007. \end{aligned}$$

Adding these all up, we obtain

$$\begin{aligned} \frac{1}{A_{2008}} - \frac{1}{A} &= 1 + 2 + \cdots + 2007 \\ &= \tfrac{1}{2}(2007)(2008) \\ &= 2015028 \end{aligned}$$

Solving for A_{2008} gives

$$A_{2008} = \frac{A}{1 + 2015028A}.$$