

Challenge of the Week

June 24–June 30, 2008

Problem

Consider a rectangular array of dots with an even number of rows and an even number of columns. Color the dots, each one red or blue, subject to the conditions that in each row half the dots are red and half are blue, and in each column half the dots are red and half are blue. Now, if two points are adjacent (in either a row or column) and like-colored, join them by an edge of their color. Show that the number of blue segments is equal to the number of red segments.

Solution

Look at two adjacent rows, A and B . Suppose row A has

- r red dots beside red dots in B ,
- b blue dots beside blue dots in B ,
- u red dots beside blue dots in B and
- v blue dots beside red dots in B .

Then there are r red and b blue segments joining elements of A to elements of B .

A has $r + u$ red dots and $b + v$ blue dots, while B has $r + v$ red dots and $b + u$ blue dots. We get that $r + u = b + v$ and $r + v = b + u$. Rearranging, we get $r - b = v - u = -(v - u)$; in other words, $r - b = 0$, or $r = b$.

Hence between adjacent rows, the number of red segments equals the number of blue segments. The same is true for adjacent columns. Thus the total number of blue segments is equal to the total number of red segments, as desired.