Challenge of the Week

July 1–July 6, 2008

Problem

Suppose that $\sin x + \cos x + \tan x + \csc x + \sec x + \cot x = 10$. Find $\sin(2x)$.

Solution

Converting everything into sines and cosines, we have

$$\begin{aligned} 10 &= \sin x + \cos x + \tan x + \cot x + \csc x + \sec x \\ &= \sin x + \cos x + \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\sin x} + \frac{1}{\cos x} \\ &= \frac{\sin^2 x \cos x + \sin x \cos^2 x + \sin^2 x + \cos^2 x + \sin x + \cos x}{\sin x \cos x} \\ &= \frac{1 + (\sin x + \cos x)(1 + \sin x \cos x)}{\sin x \cos x} \end{aligned}$$

Note that this expression involves only the unknown quantities $\sin x \cos x$ and $\sin x + \cos x$.

Let $a = \sin(2x)$ be the number we're looking for. Then we have $a = 2 \sin x \cos x$ so $\sin x + \cos x = a/2$. Also, we have $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x = 1 + a$, so that $\sin x + \cos x = \pm\sqrt{1+a}$.

Putting these into the expression above and simplifying, we find

$$\begin{aligned} 10 &= \frac{1 \pm \sqrt{1+a}(1 + a/2)}{a/2} \\ 10 &= \frac{2 \pm \sqrt{1+a}(2+a)}{a} \\ 10a - 2 &= \pm\sqrt{1+a}(2+a) \\ (10a-2)^2 &= (1+a)(2+a)^2 \\ 100a^2 - 40a + 4 &= a^3 + 5a^2 + 8a + 4 \\ a^3 - 95a^2 + 48a &= 0 \\ a(a^2 - 95a + 48) &= 0 \end{aligned}$$

So using the quadratic formula, we find $a = 0$, or $a = \frac{1}{2}\left(95 \pm \sqrt{8833}\right)$. Note that a is nonzero, as if $a = 0$, then $2 \sin x \cos x = 0$, so that either $\sin x$ or $\cos x$ is 0, which makes $\sec x$ or $\csc x$ meaningless. Next note that $a = (95 + \sqrt{8833})/2 \approx 94.5$ is impossible, since $a = \sin(2x)$ must satisfy $-1 \leq a \leq 1$. Finally, we see that $a = (95 - \sqrt{8833})/2 \approx 0.507979$ must be the solution.