Challenge of the Week

July 1–July 6, 2008

Problem

Suppose that \( \sin x + \cos x + \tan x + \csc x + \sec x + \cot x = 10 \). Find \( \sin(2x) \).

Solution

Converting everything into sines and cosines, we have

\[
10 = \sin x + \cos x + \tan x + \csc x + \sec x + \cot x
= \sin x + \cos x + \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\sin x} + \frac{1}{\cos x}
= \sin^2 x \cos x + \sin x \cos^2 x + \sin^2 x + \cos^2 x + \sin x + \cos x
= \frac{\sin x \cos x}{\sin x \cos x}

\]

Note that this expression involves only the unknown quantities \( \sin x \cos x \) and \( \sin x + \cos x \).

Let \( a = \sin(2x) \) be the number we’re looking for. Then we have \( a = 2 \sin x \cos x \) so \( \sin x + \cos x = \frac{a}{2} \). Also, we have \( (\sin x + \cos x)^2 = 1 + 2 \sin x \cos x = 1 + a \), so that \( \sin x + \cos x = \pm \sqrt{1 + a} \).

Putting these into the expression above and simplifying, we find

\[
10 = \frac{1 \pm \sqrt{1 + a} (1 + \frac{a}{2})}{\frac{a}{2}}
= \frac{2 \pm \sqrt{1 + a} (2 + a)}{a}
\]

\[
10a - 2 = \pm \sqrt{1 + a} (2 + a)
(10a - 2)^2 = (1 + a)(2 + a)^2
100a^2 - 40a + 4 = a^3 + 5a^2 + 8a + 4
a^3 - 95a^2 + 48a = 0
a(a^2 - 95a + 48) = 0
\]

So using the quadratic formula, we find \( a = 0 \), or \( a = \frac{1}{2} (95 \pm \sqrt{8833}) \). Note that \( a \) is nonzero, as if \( a = 0 \), then \( 2 \sin x \cos x = 0 \), so that either \( \sin x \) or \( \cos x \) is 0, which makes \( \sec x \) or \( \csc x \) meaningless. Next note that \( a = (95 + \sqrt{8833})/2 \approx 94.5 \) is impossible, since \( a = \sin(2x) \) must satisfy \(-1 \leq a \leq 1 \). Finally, we see that \( a = (95 - \sqrt{8833})/2 \approx 0.507979 \) must be the solution.