Challenge of the Week

July 8–July 14, 2008

Problem

A local movie theater has a contest each day. The way the contest works is each customer writes down their birthday as they buy a ticket; the first customer whose birthday matches that of an earlier customer (from the same day) gets a free ticket. Suppose that I can get in line wherever I want, but I don't know anyone else's birthday. What position in line should I take to maximize my odds of getting a free ticket?

Solution–No Leap Days

For the sake of the problem, it is reasonable to forget about leap days, and to assume that birthdays are uniformly distributed throughout the year. With this assumption, you should be the 20th person in line. (If there are fewer than 20 people, you should be last in line.)

Let $P(k)$ denote the chance of winning when you are in the kth place. Clearly, $P(1) = 0$ and $P(2) = 1/365$. To find $P(3)$, the first person can have any birthday, the second can have any birthday except one (probability 364/365) and you must match one of the first two people (probability 2/365); the product of these is $P(3) = 364 \cdot 2/365^2$.

In general,

$$\begin{aligned}
P(k) = {} & \frac{365}{365} \cdot && \text{(first person can have any birthday)} \\
& \frac{364}{365} \cdot && \text{(second person different from first)} \\
& \frac{363}{365} \cdot && \text{(third person different from first two)} \\
& \vdots && \\
& \frac{365-k+2}{365} \cdot && \text{($k-1$'st person different from all previous)} \\
& \frac{k-1}{365} && \text{(you must match one of the previous $k-1$)} \\
= {} & \frac{365 \cdot 364 \cdot 363 \cdots (365-k+2)(k-1)}{365^k}
\end{aligned}$$

One way of finding the optimal k at this point is to just compute the probabilities with a computer. We find that the probabilities increase up to $k = 20$, where the chance of winning is 3.23%, and decreases beyond that.

Alternately, we can observe that

$$\begin{aligned} P(k+1) &= \frac{365 \cdot 364 \cdot 363 \cdots (365-k+1)k}{365^{k+1}} \\ &= \frac{k(365-k+1)}{(k-1)365} P(k) \end{aligned}$$

So that $P(k+1) > P(k)$ exactly when

$$\begin{aligned} \frac{k(365-k+1)}{(k-1)365} &> 1 \\ k(365-k+1) &> 365(k-1) \\ 0 &> k^2 - k - 365 \\ 19.61 &> k \qquad \text{(solving the quadratic)} \end{aligned}$$

Again we see that P increases up to $P(19) < P(20)$, but decreases beyond this.

Solution–Accounting for Leap Days

(Due to Stefen Sharkansksy)

If we assume for this exercise that every birthday is equally likely (taking leap days into account) then the optimal place in line is 20th if you were born on regular day and 21st if you were born on a leap day.

In general, there are 366 possible birthdays. Let $P(d, k)$ be the probability that a person with birthday d wins the contest by being the kth ticket buyer. We wish to find the k which maximizes $P(d, k)$.

$P(d, k)$ is equal to the probability that the first $k-1$ ticket buyers all have different birthdays, and exactly one of them has birthday d. Assuming that every birthday is equally likely, the probability of a February 29 birthday is $q = 1/(4 \cdot 365 + 1)$ and the probability of being born on any other day is $p = 4/(4 \cdot 365 + 1)$.

Suppose first that we were born on the leap day February 29th ($d = L$). Then $P(L, 1) = 0$ and $P(L, 2) = q$. To compute $P(L, k)$ for $k \geq 3$, exactly one of the first $k-1$ ticketbuyers was born on L with probability q, the other $k-2$ were born on different days. There are $\binom{365}{k-2}$ ways to select those birthdays, with each such birthday having probability p. There are $(k-1)!$ ways to arrange the first $k-1$ buyers. So we get

$$P(L, k) = (k-1)! \binom{365}{k-2} p^{k-2} q$$

This is maximized when $k = 21$.

Suppose next that we were born on any regular day ($d = R$). Then $P(R, 1) = 0$ and $P(R, 2) = p$. To compute $P(R, k)$ for $k \geq 3$, we must consider two separate cases: (a) none of the first $k-1$ ticketbuyers was born on Feb. 29, and (b) that one of them was.

For case (a), one person was born on the same day as you, leaving 364 other non-leap days for the other $k-2$. There are $(k-1)!$ ways to arrange the birthdays for the first $k-1$ people. The probability in this case is

$$(k-1)!\binom{364}{k-2}p^{k-1}.$$

Note that this computation is exactly the probablility found in the non-leap day solution.

For case (b), one person has birthday February 29, and one matches your birthday; there are $\binom{364}{k-3}$ ways to select the other birthdays. There are $(k-1)$ ways to arrange these birthdays, and the probability of all this happening is then:

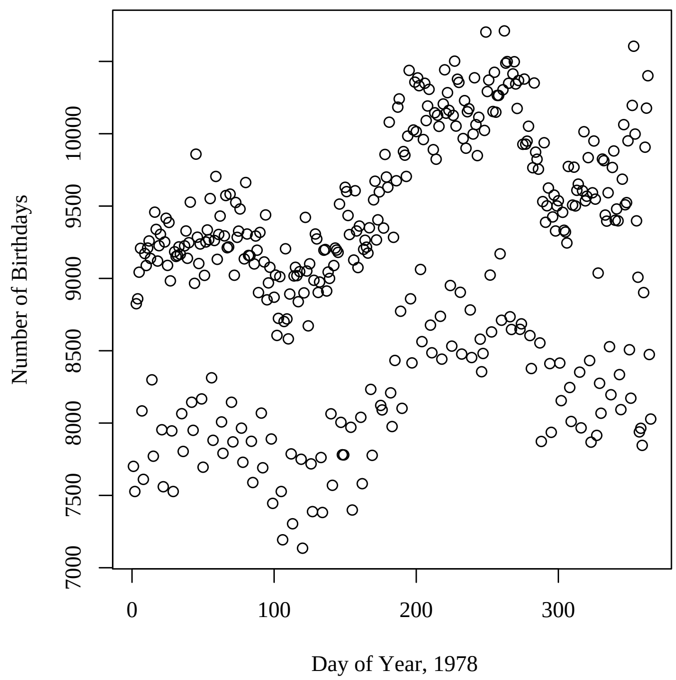
$$(k-1)!\binom{364}{k-3}p^{k-2}q.$$

So we get:

$$\begin{aligned} P(R,k) &= (k-1)!\binom{364}{k-2}p^{k-1} + (k-1)!\binom{364}{k-3}p^{k-2}q \\ &= (k-1)!p^{k-2}\left[\binom{364}{k-2}p + \binom{364}{k-3}q\right] \end{aligned}$$

In this case, $P(R,k)$ is maximized at $k=20$, which matches the non-leap day solution.

Several people pointed out that actually birthdays are *not* uniformly distributed. Konrad Shroder pointed out the website `http://www.dartmouth.edu/~chance/teaching_aids/data/birthday.txt` which contains data on birthdays in 1978. A quick look at the data shows seasonal patterns, and that there are more birthdays on week days than on weekends! For your amazement, here's a plot of the distribution:



It would be possible to use this data to solve the problem in full gory detail, but I'll leave that as an exercise for the enthusiastic reader.