

Challenge of the Week

July 15–July 21, 2008

Problem

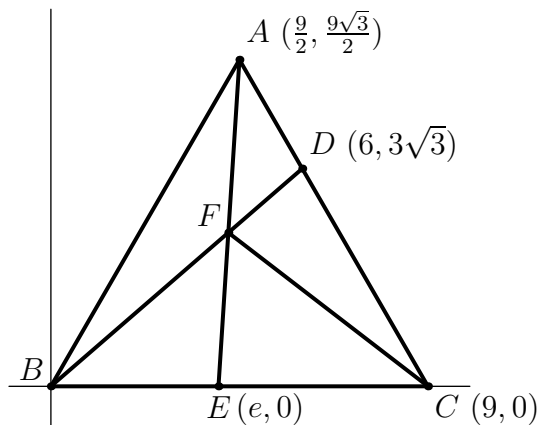
In Hughmoar County, residents are allowed to build a straight road between two homes as long as the new road is not perpendicular to any (even a non-intersecting) existing county road.

Amy, Bert, and Charles were the original residents; they lived at the corners of an equilateral triangle of side length 9 miles. They built three straight dirt roads along the sides of the triangle. Next, David built a house on the road between Amy and Charles, three miles away from Amy. Earl then built a house on the road between Bert and Charles. Two new straight roads soon were built, one between David and Bert, and another between Earl and Amy. A newcomer named Felton, who liked to be in the center of things, built his house on the intersection of these two new roads, and then made another new road straight from his house to Charles. He was promptly arrested for building a perpendicular road.

Where must Earl have built his house for this to have happened?

Solution

It's useful to draw a picture of the situation and impose a coordinate system. Put B at $(0,0)$, C at $(9,0)$, and A at $(\frac{9}{2}, \frac{9\sqrt{3}}{2})$. D must have then been built at $(6, 3\sqrt{3})$. (To see this, observe that D is $1/3$ of the way from A to C ; so the coordinates are found by the weighted average $\frac{2}{3}A + \frac{1}{3}C$.) E is on the x axis, say at $(e, 0)$.



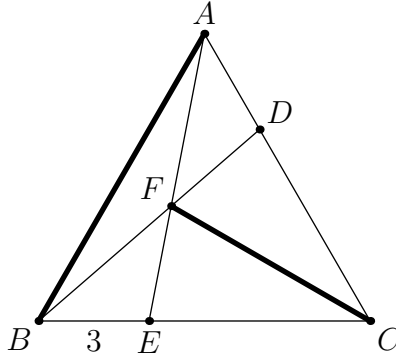
From here, F lies on the intersection of the lines AE and BD . Line AE has slope $\frac{9\sqrt{3}}{9-2e}$; line BD has slope $\frac{\sqrt{3}}{2}$. Solving

$$y - 0 = \frac{9\sqrt{3}}{9-2e}(x - e) \quad \text{and} \quad y - 0 = \frac{\sqrt{3}}{2}(x - 0)$$

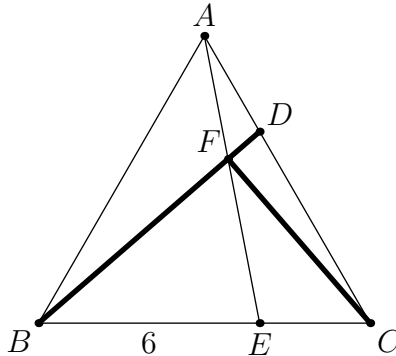
we can find the intersection F to be at $\left(\frac{18e}{9+2e}, \frac{9\sqrt{3}e}{9+2e}\right)$.

Line FC has slope $-\frac{\sqrt{3}e}{9}$, so any line perpendicular to FC has slope $\frac{9}{\sqrt{3}e}$. From the picture, there are three ways that road FC could be illegal: it could be perpendicular to AB , to BD , or to AE . We thus get three solutions:

- $FC \perp AB$. Line AB has slope $\sqrt{3}$, so solving $\frac{9}{\sqrt{3}e} = \sqrt{3}$ gives $e = 3$.



- $FC \perp BD$. Line BD has slope $\frac{\sqrt{3}}{2}$, so solving $\frac{9}{\sqrt{3}e} = \frac{\sqrt{3}}{2}$ gives $e = 6$.



- $FC \perp AE$. Line AE has slope $\frac{9\sqrt{3}}{9-2e}$; solving $\frac{9}{\sqrt{3}e} = \frac{9\sqrt{3}}{9-2e}$ gives $e = \frac{9}{5} = 1.8$.

