Challenge of the Week

July 22–July 28, 2008

Problem

The 5-12-13 triangle is special: its area and perimeter are equal. How many triangles are there with integral sides that have the same area as perimeter? (The triangles need not be right.)

Solution

Suppose the triangle has sides a, b, and c. By Heron's formula, the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$ is the semiperimeter. So we want to find integer solutions to:

$$\sqrt{s(s-a)(s-b)(s-c)} = 2s$$

which simplifies to

$$(s-a)(s-b)(s-c) = 4s.$$

We next make the clever substitutions $A = s-a$, $B = s-b$ and $C = s-c$, so that $s = A+B+C$, to get

$$ABC = 4(A+B+C).$$

Observe that A, B, and C must be positive integers. (If the semiperimeter s is an integer, then this is clear. Otherwise, s is half an odd integer, so that A, B and C are all halves of odd integers as well, meaning that ABC is not an integer, but that $4(A+B+C)$ is an integer, which is impossible.)

Assume without loss of generality that $A \leq B \leq C$. Then $ABC = 4(A+B+C) \leq 4(C+C+C) = 12C$, so that $AB \leq 12$. So we have a finite number of possibilities for A and B. Knowing A and B we can compute $C = \dfrac{A+B}{AB-4}$. The possibilities are listed below:

A	B	C		s	a	b	c
1	1	$8/(-3)$	impossible				
1	2	$12/(-2)$	impossible				
1	3	$16/(-1)$	impossible				
1	4	$20/0$	impossible				
1	5	$24/1$	$= 24$	30	29	25	6
1	6	$28/2$	$= 14$	21	20	15	7
1	7	$32/3$	impossible				
1	8	$36/4$	$= 9$	18	17	10	9
1	9	$40/5$	$= 8$ old				
1	10	$44/6$	impossible				
1	11	$48/7$	impossible				
1	12	$52/8$	impossible				
2	2	$16/0$	impossible				
2	3	$20/2$	$= 10$	15	13	12	5
2	4	$24/4$	$= 6$	12	10	8	6
2	5	$28/6$	impossible				
2	6	$32/8$	$= 4$ old				
3	3	$24/5$	impossible				
3	4	$28/8$	impossible				

So there are exactly 5 such triangles.