

# Challenge of the Week

July 29–August 4, 2008

## Problem

Place the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in the ten blank spaces in such an order that the indicated division will have a remainder of 2008.

$$3168 \overline{) 71\_543\_2\_985\_2\_356\_8\_7\_836\_7\_49528}$$

## Solution

Any arrangement will work! There are  $10!$  possible arrangements.

## Proof 1

Let  $N$  be the 34-digit dividend in the problem.  $N$  has a remainder of 2008 exactly when  $N - 2008$  has a remainder of 0, i.e., when  $N - 2008$  is divisible by 3168.

The claim is that  $N - 2008 = 71\_543\_2\_985\_2\_356\_8\_7\_836\_7\_49520$  is always divisible by 3168. Since  $3168 = 2^5 \cdot 3^2 \cdot 11$ , it suffices to check divisibility by 32, by 9, and by 11.

- $N - 2008$  is divisible by 32 since the last five digits, 47520, are divisible by 32.
- $N - 2008$  is divisible by 9 because the sum of the digits is divisible by 9. (The sum of the known digits is 117; the sum of the blanks is  $0 + 1 + \dots + 9 = 45$ ; the total is 162.)
- A number is divisible by 11 if the sum of the even numbered digits minus the sum of the odd numbered digits is divisible by 11. Here, the sum of the even numbered digits is 81; the sum of the odd numbered digits is  $7 + 4 + 8 + 5 + 3 + 7 + 2 + (0 + 1 + 2 + \dots + 9) = 81$ . The difference, 0, is divisible by 11.

## Proof 2

Define the number  $M = 7105430209850203560807083607049528$ , obtained by replacing the blanks in the dividend by 0's. Let  $a_0$  to  $a_9$  be the 10 unknown digits, so that we can write the dividend  $N$  as

$$N = M + a_0 \cdot 10^{31} + a_1 \cdot 10^{27} + a_2 \cdot 10^{25} + a_3 \cdot 10^{21} + a_4 \cdot 10^{19} + \\ a_5 \cdot 10^{15} + a_6 \cdot 10^{13} + a_7 \cdot 10^{11} + a_8 \cdot 10^7 + a_9 \cdot 10^5$$

If we look at this mod 3168 we get  $M = 568 \pmod{3168}$ , and  $10^5, 10^7, \dots, 10^{31} = 1792 \pmod{3168}$ . So,

$$N = 568 + 1792(a_0 + a_1 + \dots + a_9) \pmod{3168}$$

But for any assignment we make to  $a_0$  through  $a_9$ , the sum must be  $a_0 + a_1 + \dots + a_9 = 0 + 1 + 2 + \dots + 9 = 45$ . Thus  $N = 568 + 1792 \cdot 45 = 2008 \pmod{3168}$ .