Challenge of the Week

August 5–August 11, 2008

Problem

You go into a store, buy three things, and the cashier informs you that the cost from multiplying the prices is exactly $5.49. "That's silly," you say, "you should add the prices, not multiply them!" He tells you that it makes no difference; the sum is also $5.49.

What were the prices of the things you bought?

(If you want a harder challenge, repeat the problem with four items for $6.78. Can you find a problem where there is a unique solution involving 5 differently priced items?)

Solution

Converting everything into cents (so that we can look for integer solutions) we want to solve

$$a + b + c = 549$$
$$abc = 5490000$$

Notice that $5490000 = 2^4 \cdot 5^4 \cdot 3^2 \cdot 61$, so one of the numbers must be divisible by 61. Without loss of generality, suppose that c is a multiple of 61. Note that c must be less than $9 \cdot 61 = 549$ so that the other items can have positive price. Also c cannot be $7 \cdot 61$ since 5490000 is not divisible by 7. There are only a few possible values for c; given a particular value, we can easily find $a+b$ and ab:

c	$a+b$	ab
$1 \cdot 61 = 61$	488	90000
$2 \cdot 61 = 122$	427	45000
$3 \cdot 61 = 183$	366	30000
$4 \cdot 61 = 244$	305	22500
$5 \cdot 61 = 305$	244	18000
$6 \cdot 61 = 366$	183	15000
$8 \cdot 61 = 488$	61	11250

Some of these cases we can remove immediately using the following observation: any two positive numbers a and b must satisfy $\sqrt{ab} \le \frac{a+b}{2}$; this is the arithmetic mean-geometric mean inequality. Only $c \in \{122, 183, 244\}$ satisfy this; the other cases are impossible.

For the remaining three possibilities, we can just solve a pair of equations to get a and b. E.g., if $c = 122$, then solve $a + b = 427, ab = 45000$. The only case we get integral solutions for a and b is when $c = 244$, $a = 125$ and $b = 180$.

So, up to permutation, the prices are $1.25, $1.80, and $2.44.

For the harder challenges: using similar but more involved arguments, it's possible to show that there is a unique solution to the four-item problem: \$6.78 = \$1.13 + \$1.25 + \$2.00 + \$2.40.

The easiest way to find and solve these problems is with a computer search. There are lots of problems, but not so many that you're likely to stumble upon one by accident. Here are the smallest problems (i.e., with the smallest total) where the solution is unique and the prices are distinct:

number of items	problem
2	\$4.05 = \$1.80 + \$2.25
3	\$5.25 = \$1.50 + \$1.75 + \$2.00
4	\$6.44 = \$1.25 + \$1.60 + \$1.75 + \$1.84
5	\$7.62 = \$1.25 + \$1.27 + \$1.50 + \$1.60 + \$2.00
6	\$8.88 = \$1.00 + \$1.25 + \$1.28 + \$1.50 + \$1.85 + \$2.00
7	\$13.25 = \$0.25 + \$1.00 + \$1.25 + \$1.60 + \$2.50 + \$2.65 + \$4.00
8	\$22.58 = \$0.40 + \$0.50 + \$0.64 + \$1.00 + \$1.25 + \$2.50 + \$5.00 + \$11.29