

Challenge of the Week

August 12–August 18, 2008

Problem

All of an 8×8 chessboard, with the exception of one square, is covered by 1×3 rectangular tiles. How many of the 64 squares can occur as the uncovered square?

(For a slightly harder problem, consider tiling a 9×9 board with 1×4 tiles. Which of the 81 squares can occur as the uncovered square?)

8×8 Solution

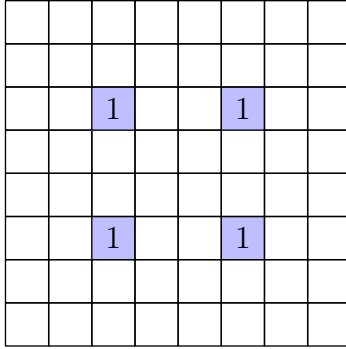
There are four such squares. We first show that at most four squares are possible, using a parity argument. Consider the following two labelings of the squares of the chessboard:

0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0
0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0
0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2

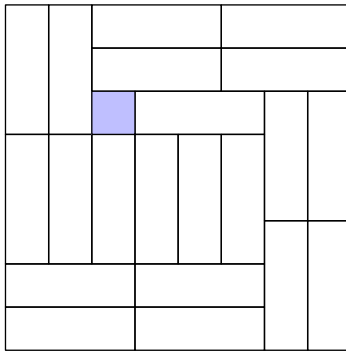
1	0	2	1	0	2	1	0
2	1	0	2	1	0	2	1
0	2	1	0	2	1	0	2
1	0	2	1	0	2	1	0
2	1	0	2	1	0	2	1
0	2	1	0	2	1	0	2
1	0	2	1	0	2	1	0
2	1	0	2	1	0	2	1

(I've colored the squares to make the patterns of the labelings easier to see; the '1's are the most important.) There are two key properties of each of these labelings: first, any 1×3 rectangle covers up a '0', '1', and '2'; second, there are 22 '1's, but only space for 21 rectangles.

Because of these properties, any way we tile the board with 1×3 rectangles, the leftover square must be a '1'. Since this must work for both the above labelings, the only possible places for the leftover square are where there are '1's common in both labelings. There are at most four possible squares:

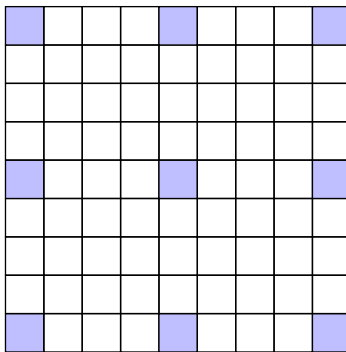


To complete the solution, we must show that there are indeed tilings where these squares are not covered. Here's a tiling for the upper-left square; rotations of this tiling give examples for the other three squares.



9 × 9 Solution

With not much work, we can find tilings with 4×1 rectangles where any one of the squares in the picture below is missing.



To show that indeed there are no other possibilities, we can use a parity argument just as in the 8×8 problem—it just takes a little more work to find a set of appropriate labelings. The set of two labelings below works: any 1×4 rectangle covers up a '0', '1', '2', and '3'; and there is one more '0' than any other number, so any tiling must have the missing square where there is a '0' common to both the labelings below.

0	1	2	3	0	1	2	3	0
1	2	3	0	1	2	3	0	1
2	3	0	1	2	3	0	1	2
3	0	1	2	3	0	1	2	3
0	1	2	3	0	1	2	3	0
1	2	3	0	1	2	3	0	1
2	3	0	1	2	3	0	1	2
3	0	1	2	3	0	1	2	3
0	1	2	3	0	1	2	3	0

0	1	2	3	0	1	2	3	0
2	3	0	1	2	3	0	1	2
3	0	1	2	3	0	1	2	3
1	2	3	0	1	2	3	0	1
0	1	2	3	0	1	2	3	0
2	3	0	1	2	3	0	1	2
3	0	1	2	3	0	1	2	3
1	2	3	0	1	2	3	0	1
0	1	2	3	0	1	2	3	0