Challenge of the Week

August 19–August 25, 2008

Problem

Find all integers x, y, and z so that $x^3 + y^3 + z^3 = x + y + z = 3$.

Solution 1

Subsituting $x = 3 - y - z$ into the equation $x^3 + y^3 + z^3 = 3$ yields the following quadratic equation in y:

$$(3 - z)y^2 - (z^2 - 6z + 9)y + (3z^2 - 9z + 8) = 0.$$

From the quadratic formula we see

$$y = \frac{(z^2 - 6z + 9) \pm \sqrt{(z^2 - 6z + 9)^2 - 4(3 - z)(3z^2 - 9z + 8)}}{2(3 - z)},$$

which simplifies to

$$y = \frac{(z - 3)^2 \pm (z - 1)\sqrt{(z - 3)(z + 5)}}{2(3 - z)}.$$

For y to be an integer, we must have $(z-1)\sqrt{(z - 3)(z + 5)}$ be an integer; this requires either that $z = 1$ or that $(z - 3)(z - 5)$ is a perfect square.

If we suppose that $z = 1$, then we immediately get the solution $(x, y, z) = (1, 1, 1)$.

Otherwise, setting $(z - 3)(z + 5) = k^2$ (assuming, without loss of generality, that $k \geq 0$) yields $z = -1 \pm \sqrt{16 + k^2}$, so that $16 + k^2$ is a perfect square. Supposing that $16 + k^2 = m^2$, we can rewrite this as $(k + m)(k - m) = 16$, so that $k + m$ and $k - m$ are factors of 16. trying the various factorizations and solving for k and m yield two non-negative integer solutions: $k = 0$ and $k = 3$.

Putting in these values for k yields $z = 4, -5, 3$ or -6. Neither $z = 3$ nor $z = -6$ work since they both give a non-integer answer for y. When $z = 4$, we get the solutions $(x, y, z) = (-5, 4, 4)$ and $(4, -5, 4)$; when $z = -5$ we get the solution $(x, y, z) = (4, 4, -5)$.

Solution 2

Since this problem involves the quantities $x+y+z$ and $x^3+y^3+z^3$, we first search for a relationship between them. If we cube $x+y+z$, we get

$$(x+y+z)^3 = x^3+y^3+z^3+3(x^2y+x^2z+y^2z+xy^2+xz^2+yz^2)+6xyz,$$

which factors as:

$$(x+y+z)^3 = (x^3+y^3+z^3)+3(x+y)(x+z)(y+z).$$

After this trickery, we can substitute $x^3+y^3+z^3=3$, $x+y+z=3$, and $x+y=3-z$, etc., to get:

$$3^3 = 3+3(3-z)(3-y)(3-x)$$

Dividing by 3 and simplifying signs then gives

$$-8 = (x-3)(y-3)(z-3).$$

We find that $x-3$, $y-3$, and $z-3$ must be factors of -8. Without loss of generality, assume that $x \le y \le z$; then writing the possible factorizations of -8 (with the factors in increasing order) we find all the possiblities for x, y, and z:

factorization			x	y	z	$x+y+z$
-8	-1	-1	-5	2	2	-1
-8	1	1	-5	4	4	3 ✓
-4	-2	-1	-1	1	2	2
-4	1	2	-1	4	5	8
-2	-2	-2	1	1	1	3 ✓
-2	1	4	1	4	7	12
-2	2	2	1	5	5	11
-1	1	8	2	4	11	17
-1	2	4	2	5	7	14

The only cases that work are $(x,y,z)=(1,1,1)$ and $(-5,4,4)$ (and permutations thereof).