Challenge of the Week

August 26–September 8, 2008

Problem

During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise around the children and hands out candy to some of them according to the following procedure:

1. Start by picking a child and give them a piece of candy
2. Skip 1 child and give a piece of candy to the next
3. Skip 2 children, give a piece of candy to the next
4. Skip 3 children, give a piece of candy to the next
5. etc.

Determine the values of n for which all children will eventually receive some candy.

Solution

All the children get candy if and only if n is a power of two.

Let $T(k) = k(k+1)/2$ be the kth triangular number. Associating the children to the integers modulo n, the kth piece of candy is given to the child $T(k) \bmod n$. The goal of this problem is to find n so that $T \pmod{n}$ is surjective. (In other words, we want to find n so that there are integers $k_0, \ldots, k_{n-1}$ with $T(k_0), \ldots, T(k_{n-1}) \equiv 0, \ldots, n-1 \pmod{n}$.)

Claim: If n is not a power of 2, then $T \pmod{n}$ is not surjective—some children never get candy.

Proof: Suppose (by way of contradiction) that $T \pmod{n}$ is surjective; there are candies $k_0, \ldots, k_{n-1}$ given to the children $0, \ldots, n-1$. Since n is not a power of 2, there must be an odd prime p that divides n. Consider the simpler problem of giving candy to just p children; $T \pmod{p}$ must also be surjective since the candies $k_0, \ldots, k_{p-1}$ are given to the children $0, \ldots, p-1$.

Now note that $T(k+p) = T(k) \pmod{p}$. This is easily verified in the following computation:

$$\begin{aligned} T(k+p) &= \frac{(k+p)(k+p+1)}{2} \\ &= \frac{k^2+k+p(2k+p+1)}{2} \\ &= T(k) + pk + p\frac{p+1}{2} \\ &\equiv T(k) \pmod{p}. \end{aligned}$$

The upshot of this computation is that the values of $T(0), T(1), \ldots, T(p-1)$ determine the values of T for all k; if a child receives any candy, he must in fact receive one of the first p pieces of candy. As all the children get candy, each of the first p pieces must go to a different child, i.e., $T(0), \ldots, T(p-1)$ must be distinct.

But this is not the case! We compute that $T(0) = 0$ and $T(p-1) \equiv T(-1) \equiv 0 \pmod{p}$ so we have a contradiction. We must conclude that $T \pmod{n}$ is not surjective.

Claim: If $n = 2^m$, then $T \pmod{n}$ is surjective—all the children get candy.

Proof: We begin with a short computation to show that $T(k + 2^{m+1}) \equiv T(k) + 2^m \pmod{2^{m+1}}$:

$$\begin{aligned} T(k+2^{m+1}) &= \frac{(k+2^{m+1})(k+2^{m+1}+1)}{2} \\ &= \frac{k^2 + (2\cdot 2^{m+1}+1)k + 2^{m+1}}{2} \\ &= \frac{k(k+1)}{2} + \frac{2\cdot 2^{m+1}k + 2^{m+1}}{2} \\ &= T(k) + 2^{m+1}k + 2^m \\ &\equiv T(k) + 2^m \pmod{2^{m+1}}. \end{aligned}$$

Now we prove the claim by induction on m. When $m = 0$ or 1, we have $n = 1$ or 2 children, and it's easy to see they all get candy. So $T \pmod{n}$ is surjective in these cases.

Next assume that T is surjective when $n = 2^m$, so that there are be integers $k_1, \ldots, k_{2^m}$ so that $T(k_i) \equiv i \pmod{2^m}$. We must show that $T \pmod{2^{m+1}}$ is surjective.

If we double the number of children, then $T(k_i)$ is equivalent to either i or to $i + 2^m \pmod{2^{m+1}}$; from the computation, the teacher will give candy to both children i and $i + 2^m$ with the k_i-th and $(k_i + 2^m)$-th pieces. Then the pieces of candy $k_1, \ldots, k_{2^m}, k_1 + 2^{m+1}, \ldots, k_{2^m} + 2^{m+1}$ are enough to give each of the 2^{m+1} children one piece of candy each; we conclude $T \pmod{2^{m+1}}$ is surjective, completing the proof.